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OF THE

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F. W. DYSON, Esq., M.A., F.R.S., PRESIDENT, in the Chair.

Sydney Chapman, Royal Observatory, Greenwich, S.E.; and
Harold N. Haskell, B.A., Mathematical Master, Harrison School, Barbados, West Indies,

were balloted for and duly elected Fellows of the Society.

The following candidates were proposed for election as Fellows of the Society, the names of the proposers from personal knowledge being appended:—

The Rev. William Aitken Haslam, M.A., 19 Dalebury Road, Wandsworth Common, S.W. (proposed by S. A. Saunder);
Evan Lewis Jones, B.Sc., Class-Teacher, 12 Ladysmith, Troedyrhin, Merthyr-Tydfil, South Wales (proposed by H. Shaw); and
Henry Morris, B.Sc. (Lond.), Senior Science Master, Technical and County Secondary School, Beckenham, and 44 Belmont Road, Beckenham, Kent (proposed by A. Fowler).

One hundred and ten presents were announced as having been received since the last ordinary meeting, including amongst others:—

Œuvres Complètes de Christiaan Huygens, tome 12, presented by the Amsterdam Academy; three volumes of Annales de l'Observatoire de Paris (Mémoires), and the 12th. and concluding part of Loewy and Puiseux, Atlas photographique de la lune, presented by the Paris Observatory; M. Jarry-Desloges, Observa-

tions des surfaces planétaires, fasc. 2, 1909–10, presented by the Author; C. E. Monro, on the portable reversible Transit Instrument, presented by the Author; A. P. Norton, Star Atlas, presented by the Publishers.

Astrographic Chart of the Heavens: 59 charts from the Bordeaux and Toulouse Observatories, presented by the French Minister of Public Instruction, and 12 charts presented by the Tacubaya Observatory; original negative of the total solar eclipse of 1910 May 10, taken by Mr. Sargeant at Queenstown, Tasmania, presented by Mr. A. A. Cox; Photographs of Mars and Nebulæ (10 transparencies and 5 lantern slides), presented by Prof. E. E. Barnard; Photograph from a drawing of Halley's Comet, presented by Mr. Maxwell Hall.

On the bearing of the Principle of Relativity on Gravitational Astronomy. By W. de Sitter, Assoc. R.A.S.

1. The principle of relativity, first developed in connection with the electromagnetic theory of light, has in recent years been more and more considered as of universal application, and the claim has been made that the whole of our physical sciences should be framed in conformity with it. In this paper are considered some of the results of introducing the principle in the laws of planetary motion. Poincaré has made the remark that some of the difficulties of the problem of three bodies are due to the fact that in the undisturbed motion the nodes and perihelia are fixed. Now, with a law of attraction which is in conformity with the principle of relativity, the perihelion has a motion even in the problem of two bodies, and it is probable that such questions as the existence of periodic solutions and the convergence of series entirely change their aspect. In the present paper, however, these interesting mathematical questions are not touched upon, and I consider the problem exclusively from the point of view of the practical astronomer, investigating only such effects as may be expected to yield the possibility of an empirical verification of the principle.

All natural phenomena are described in terms of four variable quantities: three "coordinates" x, y, z, and the "time" t. If we introduce other variables x', y', z', t', the formulæ describing any event will generally be altered. Also the physical laws, which are differential equations defining x, y, z, as functions of t and certain constants, are in general altered by a change of the system of reference. Physical science has, however, been so built up that there are certain transformations which leave the form of the equations unaltered. In the classical, or Newtonian, mechanics these are all orthogonal transformations of the three coordinates x, y, z, combined with an arbitrary uniform velocity of these axes and an arbitrary change of the units of length and time.

The principle of relativity can be enunciated as the postulate

that the transformations, with respect to which the laws of nature shall be invariant, are "Lorentz-transformations." * A Lorentz-transformation is defined by a modulus q and an axis. Taking the latter as the axis of x, the transformation-formulæ are—

$$x' = \frac{x - qct}{\sqrt{1-q^2}}, \quad y' = y, \quad z' = z, \quad ct' = \frac{ct - qx}{\sqrt{1-q^2}},$$

where c is a universal constant, which, according to the electromagnetic theory, is equal to the velocity of light in free space. In Newtonian mechanics the value of this constant is $c = \infty$. Putting, then, $q = v/c$, so that $q = 0$, the Lorentz-transformation degenerates into a "Newton-transformation,"—

$$x' = x - vt, \quad y' = y, \quad z' = z, \quad t' = t.$$

To both Lorentz- and Newton-transformations may be added an arbitrary orthogonal transformation of coordinates.

The physical meaning of the principle has been very clearly explained in these pages by Messrs. Plummer and Whittaker,† and need not be repeated here. The mathematical formulæ are all that is required for our purpose.

The literature of the subject is very extensive, and it is hardly possible for an outsider to be even superficially acquainted with it. Also I do not claim originality for any of the formulæ or results given below. The starting-point of my investigations has been the papers by Poincaré and Minkowski.‡ The manner in which the equations of motion are derived below is entirely derived from the last section of Poincaré's paper. I also owe much to conversations with and advice from my colleague Professor Lorentz.

2. Let there be two systems of reference:—
the "general" system (x', y', z', t'),
and the "special" system (x, y, z, t).
The first is an absolutely arbitrary system of reference. The

* This name has been first introduced by Poincaré in the paper quoted below.

† *H. C. Plummer:* "On the Theory of Aberration and the Principle of Relativity" (*M.N.*, Jan. 1910). *E.T.W.:* "Recent Researches on Space, Time, and Force" (Report of the Council, *M.N.*, Feb. 1910). Both authors make free use of the word "aether." As there are many physicists nowadays who are inclined to abandon the aether altogether, it may be well to point out that the principle of relativity is essentially independent of the concept of an aether, and, indeed, is considered by some to lead to a negation of its existence. Astronomers have nothing to do with the aether, and it need not concern them whether it exists or not. All Mr. Plummer's results remain true, and retain their full value, if the "aether" is eliminated from his terminology. And also in Mr. Whittaker's note the word "aether" is not essential, except, of course, from an historical point of view.

‡ *Poincaré:* "Sur la dynamique de l'électron," *Rendiconti del circolo matematico di Palermo*, vol. xxi. p. 129 (Dec. 1905).

Minkowski: "Die Grundgleichungen für die electromagnetische Vorgänge in bewegten Körpern," *Göttinger Nachrichten*, Math. physik. Klasse, 1908, page 53.

three variables x', y', z' define what Newton would call absolute space, and t' corresponds to absolute time.

The second is a system of reference of which the origin and the direction of the axes are chosen according to the needs of the problem in hand. The transformation from x, y, z, t to x', y', z', t', or inversely, is a Lorentz-transformation.

The chief consequence of the principle of relativity is that it is impossible to observe any but relative motions. The "general" system only differs from any other possible system in that no convention is made as to its origin or the direction of its axes. It is thus not possible, from the point of view of the principle of relativity, to speak of "absolute" velocity or position otherwise than as velocity or position relative to any, not specified, "general" system of reference. If the word "absolute" is used in this sense, it is unobjectionable, but unnecessary. The laws of nature must, of course, be primarily framed with respect to the general system of reference. This does not mean that they assert anything about "absolute" motion or "absolute" time, but only that they must be so built up as to be true in whatever system of reference we choose to use.

We have thus to consider two problems:—

(a) What is the law of force that must replace Newton's law, and what is the motion of a planet under this law? So far as this differs from ordinary Keplerian motion, we shall have to consider the question whether the differences are large enough to be verified by observation.

(b) A motion being given in any "special" system, how is it expressed in the "general" system, or in any other special system? This only involves the working out of the formulæ of transformation for the required cases.

A third question then arises, viz. whether the "astronomical system," *i.e.* the coordinates and the time actually used by practical astronomers, coincides or not with any "special" system as defined above. This question is practically not affected by the introduction of the principle of relativity, and we need refer to it only very briefly.

3. I will now begin by stating the necessary formulæ for a Lorentz-transformation in the form which I find most convenient for my purpose. Very probably most of them have already been published elsewhere in the same form, but I found it easier to work them out for myself than to search for them in an unfamiliar literature.

The formulæ become more symmetrical if the unit of time is so chosen that $c=1$. From an astronomical point of view, however, it is more convenient to retain the ordinary unit of time. Both advantages are combined by using ct as variable instead of t. Derivatives with respect to ct are denoted by Greek letters, thus—

$$\xi=\frac{dx}{cdt},\quad \eta=\frac{dy}{cdt},\quad \zeta=\frac{dz}{cdt}.$$

Further, we put

$$r^2 = x^2 + y^2 + z^2, \qquad \phi^2 = \xi^2 + \eta^2 + \zeta^2.$$

The modulus of the tranformation is q. We also introduce $q_1 = 1 - \sqrt{1 - q^2}$. If q is a small quantity of the first order, then q_1 is of the second order. The cosines of the angles which the axis of the transformation makes with the axes of x, y, z are denoted by α, β, γ, so that $\alpha^2 + \beta^2 + \gamma^2 = 1$.

The transformation-formulæ are then—

$$\begin{aligned} x' &= x + \frac{\alpha[q_1 r_q - qct]}{\sqrt{1 - q^2}}, \\ y' &= y + \frac{\beta[q_1 r_q - qct]}{\sqrt{1 - q^2}}, \\ z' &= z + \frac{\gamma[q_1 r_q - qct]}{\sqrt{1 - q^2}}, \\ ct' &= \frac{ct - qr_q}{\sqrt{1 - q^2}}. \end{aligned} \tag{1}$$

We find easily

$$\xi' = \frac{dx'}{cdt'} = \frac{\xi\sqrt{1 - q^2} + \alpha[q_1\phi_q - q]}{1 - q\phi_q}, \tag{2}$$

and similarly for η' and ζ'.

Further,

$$1 - \phi'^2 = \frac{(1 - \phi^2)(1 - q^2)}{(1 - q\phi_q)^2}.$$

In these formulæ r_q and ϕ_q are the projections of r and ϕ on the axis of transformation :—

$$\begin{aligned} r_q &= \alpha x + \beta y + \gamma z, \\ \phi_q &= \alpha\xi + \beta\eta + \gamma\zeta. \end{aligned}$$

If we put

$$(\kappa) = \frac{1}{\sqrt{1 - \phi^2}},$$

$$(\xi) = \frac{\xi}{(\kappa)}, \quad (\eta) = \frac{\eta}{(\kappa)}, \quad (\zeta) = \frac{\zeta}{(\kappa)},$$
$$(\phi)^2 = (\xi)^2 + (\eta)^2 + (\zeta)^2,$$

we can easily verify that the transformation-formulas for (ξ), (η), (ζ), (κ) are the same as those for x, y, z, t, viz.—

$$\begin{aligned} (\xi)' &= (\xi) + \frac{\alpha[q_1(\phi)_q - q(\kappa)]}{\sqrt{1 - q^2}}, \\ (\kappa)' &= \frac{(\kappa) - q(\phi)_q}{\sqrt{1 - q^2}}. \end{aligned} \tag{3}$$

The invariants of the transformation are all of the form—

$$x_1x_2 + y_1y_2 + z_1z_2 - ct_1.ct_2\,, \tag{4}$$

where x_1, y_1, z_1, ct_1, or x_2, y_2, z_2, ct_2 may be replaced by any set of quantities, which are transformed by the same formulæ, such as (ξ), (η), (ζ), (κ), etc.

The equation

$$(\kappa)^2 - (\phi)^2 = 1$$

is thus not altered by the transformation. If now we define a new variable $c\tau$ by the equations

$$\frac{d\tau}{dt} = \sqrt{1-\phi^2} = \frac{1}{(\kappa)}\,, \quad \text{or} \quad \frac{dt}{d\tau} = (\kappa) = \sqrt{1+(\phi)^2}\,, \tag{5}$$

this variable is the same function of x', y', z', t' as of x, y, z, t, and is consequently independent of the system of reference. We have, of course,

$$(\xi) = \frac{dx}{cd\tau}\,, \quad (\eta) = \frac{dy}{cd\tau}\,, \quad (\zeta) = \frac{dz}{cd\tau}\,, \quad (\kappa) = \frac{dct}{cd\tau}\,.$$

The variable τ is called by Minkowski the "Eigenzeit" of the point whose coordinates are x, y, z, which may be translated by "proper-time."* In many problems it is more convenient as an independent variable than t.

Every point has thus its own proper-time, which is independent of the system of reference, but depends on the state of motion of the point and on its previous history. The proper-time of a point rigidly connected with the axes of the system of reference (x, y, z, t) is t itself. As a convenient abbreviation, we may speak of "heliocentric time," "geocentric time," etc., meaning the proper-time of the Sun, the Earth, etc.

4. A set of values of m, x, y, z, t, defining the position of a particle of mass m in the system of reference (x, y, z, t), may be called an "event." Two events are called simultaneous if their values of t are the same. Two events which are simultaneous in one system (x, y, z, t) are in general not simultaneous in another system (x', y', z', t'). And, within certain restrictions, which are of no importance for our purpose, a system can always be found in which two arbitrarily given events are simultaneous.

We have

$$c(t'_1 - t'_2) = \frac{c(t_1 - t_2) - q\Delta_q}{\sqrt{1-q^2}}\,,$$

where

$$\Delta_q = \alpha(x_1 - x_2) + \beta(y_1 - y_2) + \gamma(z_1 - z_2)\,.$$

* It should be remarked that this is *not* the same as "local" time, as originally defined by Lorentz.

Thus, if we wish the two events to be simultaneous with respect to the time t', or $t'_1 = t'_2$, we have

$$c(t_1 - t_2) = q\Delta_q .$$

Denoting the coordinates of simultaneous events by non-italic letters, we find thus—

$$\mathrm{x}'_1 - \mathrm{x}'_2 = x_1 - x_2 + \frac{\alpha[q_1\Delta_q - qc(t_1 - t_2)]}{\sqrt{1 - q^2}},$$

or, remarking that $q_1 - q^2 = -q_1\sqrt{1 - q^2}$,

$$\mathrm{x}'_1 - \mathrm{x}'_2 = x_1 - x_2 - \alpha q_1\Delta_q , \qquad (6)$$

and similarly for the other coordinates.

We find easily, denoting the distance between simultaneous positions by $\boldsymbol{r}$:—

$$\boldsymbol{r}'_q = \Delta'_q = \Delta_q - q_1\Delta_q(\alpha^2 + \beta^2 + \gamma^2) = (1 - q_1)\Delta_q = \Delta_q\sqrt{1 - q^2} .$$

Therefore (6) can be written—

$$x_1 - x_2 = \mathrm{x}'_1 - \mathrm{x}'_2 + \alpha\frac{q_1\boldsymbol{r}'_q}{\sqrt{1 - q^2}} . \qquad (7)$$

5. If we consider the action on m_1 at time t_1 of a force emanating from m_2 at time t_2, we will suppose—

$$c(t_1 - t_2) = \Delta = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2} . \qquad (8)$$

The expression

$$c^2(t_1 - t_2)^2 - \Delta^2$$

is of the general form (4), and is thus an invariant of the transformation. We have thus also $c(t'_1 - t'_2) = \Delta'$. The equation (8) states that the force is propagated through space with the velocity of light. This is, of course, an arbitrary assumption, which is not a necessary consequence of the principle of relativity. The velocity of propagation might be defined by *any* invariant of the transformation containing $c(t_1 - t_2)$, put equal to zero. But it is a natural assumption, and the most simple which can be made.*

Denoting now the *simultaneous relative* coordinates by letters of another type, $\boldsymbol{x}$, $\boldsymbol{y}$, $\boldsymbol{z}$, we have, for time t_1,

$$\boldsymbol{x} = \mathrm{x}_1 - \mathrm{x}_2 = x_1 - x_2 - \Delta\xi_2 - \delta\dot{x}_2 , \qquad (9)$$

where

$$\delta x_2 = \tfrac{1}{2}\Delta^2\frac{d^2x_2}{c^2dt^2} + \tfrac{1}{6}\Delta^3\frac{d^3x_2}{c^3dt^3} + \ldots .$$

* It is, of course, not allowed to put $c(t_1 - t_2) = 0$, which would mean instantaneous action at a distance, since the expression $c(t_1 - t_2)$ by itself is not invariant.

The values of ξ_2 and the higher differential coefficients must be taken for the time t_2, defined by (8). We consider $1/c$ as a small quantity of the first order, and we wish our formulæ to be exact to the second order inclusive.

We find easily

$$\Delta^2(1-\phi_2^2) = r^2 + 2r\epsilon_2\Delta + 2r\chi_2,$$

where

$$r\epsilon_1 = x\xi_1 + y\eta_1 + z\zeta_1,$$
$$r\epsilon_2 = x\xi_2 + y\eta_2 + z\zeta_2,$$
$$r\chi_2 = x\delta x_2 + y\delta y_2 + z\delta z_2.$$

ϵ_1 and ϵ_2 are of the first order, χ_2 is of the second order.

In the equations of motion there appear the invariants—

$$A = c(t_2-t_1)(\kappa)_1 - (x_2-x_1)(\xi)_1 - (y_2-y_1)(\eta)_1 - (z_2-z_1)(\zeta)_1,$$
$$B = c(t_1-t_2)(\kappa)_2 - (x_1-x_2)(\xi)_2 - (y_1-y_2)(\eta)_2 - (z_1-z_2)(\zeta)_2,$$
$$C = (\kappa)_1(\kappa)_2 - (\xi)_1(\xi)_2 - (\eta)_1(\eta)_2 - (\zeta)_1(\zeta)_2.$$

To express these in simultaneous relative coordinates we have—

$$\left.\begin{aligned} A\sqrt{1-\phi_1^2} &= -\Delta(1-\xi_1\xi_2-\eta_1\eta_2-\zeta_1\zeta_2) + r\epsilon_1, \\ B^2(1-\phi_2^2) &= r^2(1-\phi_2^2+\epsilon_2^2) + 2r\chi_2, \end{aligned}\right\} \quad (10)$$

$$C = \frac{1-\xi_1\xi_2-\eta_1\eta_2-\zeta_1\zeta_2}{\sqrt{(1-\phi_1^2)(1-\phi_2^2)}}.$$

6. The equations of motion can be given in three forms:—

$$mc\frac{d(\xi)}{d\tau} = m\frac{d^2x}{d\tau^2} = (X) = \frac{X}{\sqrt{1-\phi^2}}, \quad (11)$$

$$mc\frac{d(\xi)}{dt} = m\frac{d}{dt}\left(\frac{dx}{d\tau}\right) = X = (X)\sqrt{1-\phi^2}, \quad (12)$$

$$m\frac{d^2x}{dt^2} = \left\{X - \xi(X\xi + Y\eta + Z\zeta)\right\}\sqrt{1-\phi^2}, \quad (13)$$

and similarly for the other coordinates.

X is the force according to the ordinary definition, or "Newtonian" force; (X) is called the "Minkowskian" force.* The mass m is a constant.

Differentiating the formula $(\kappa)^2 = 1 + (\phi)^2$, we derive a fourth equation analogous to (11) or (12), viz.:

$$mc\frac{d(\kappa)}{d\tau} = m\frac{d^2(ct)}{d\tau^2} = (T) = \frac{T}{\sqrt{1-\phi^2}}, \quad (11')$$

* See Lorentz, Göttingen Lectures, October 1910 (*Physikalische Zeitschrift*, vol. xi. p. 1234). The names quoted above occur on page 1237.

or

$$mc\frac{d(\kappa)}{dt} = m\frac{d}{dt}\left(\frac{dct}{d\tau}\right) = \mathrm{T} = (\mathrm{T})\sqrt{1-\phi^2}\,, \qquad (12')$$

where

$$\mathrm{T} = \mathrm{X}\xi + \mathrm{Y}\eta + \mathrm{Z}\zeta\,.$$

Minkowski gives the name "kinetic energy" to the quantity

$$\mathrm{E} = mc^2[(\kappa) - 1] = \tfrac{1}{2}mc^2\phi^2 + \tfrac{3}{8}mc^2\phi^4 + \ldots\ldots,$$

or

$$\mathrm{E} = \tfrac{1}{2}m\mathrm{v}^2(1 + \tfrac{3}{4}\phi^2 + \ldots\ldots)\,.$$

The equation (12′) thus turns out to be the equation of energy,

$$\frac{d\mathrm{E}}{dt} = \mathrm{X}\frac{dx}{dt} + \mathrm{Y}\frac{dy}{dt} + \mathrm{Z}\frac{dz}{dt}\,.$$

If we use Minkowskian velocities and forces, and put

$$(\mathrm{E}) = \tfrac{1}{2}mc^2(\phi)^2,$$

we find similarly from (11′)

$$\frac{d(\mathrm{E})}{d\tau} = (\mathrm{X})\frac{dx}{d\tau} + (\mathrm{Y})\frac{dy}{d\tau} + (\mathrm{Z})\frac{dz}{d\tau}\,.$$

The law of the force must be such that the form of the equations of motion is not changed by a Lorentz-transformation. Therefore

$$(\mathrm{X}),\ (\mathrm{Y}),\ (\mathrm{Z}),\ (\mathrm{T})$$

must be transformed by the same formulæ as

$$x,\quad y,\quad z,\quad t,$$
$$(\xi),\ (\eta),\ (\zeta),\ (\kappa),$$

or, in other words, (X), (Y), (Z) must be linear functions of

$$x_1,\quad y_1,\quad z_1,\qquad x_2,\quad y_2,\quad z_2,$$
$$(\xi)_1,\ (\eta)_1,\ (\zeta)_1,\qquad (\xi)_2,\ (\eta)_2,\ (\zeta)_2,$$

the coefficients being invariants of the transformation.

For zero velocities the equations of Newtonian mechanics must be reproduced, therefore the coordinates can only enter by their differences $x_1 - x_2$, etc.

Introducing further the condition that the resulting equations must not contain the velocities in the first degree, and certain other simplifications, Poincaré is led to take (*l.c.*, page 174)—

$$(\mathrm{X})_1 = \frac{k^2 m_1 m_2}{\mathrm{B}_1^{\,3}}\left\{ x_2 - \mathrm{x}_1 - (\xi)_2\frac{\mathrm{A}_1}{\mathrm{C}}\right\}. \qquad (14)$$

Similarly we have for the force acting on m_2 from m_1,

$$(\mathrm{X})_2 = \frac{k^2 m_1 m_2}{\mathrm{A}_2^{\,3}} \left\{ x_1 - \mathrm{x}_2 - (\xi)_1 \frac{\mathrm{B}_2}{\mathrm{C}} \right\}. \tag{15}$$

We will now introduce simultaneous coordinates. Let these be for time t—

$$\mathrm{x}_1,\ \mathrm{y}_1,\ \mathrm{z}_1, \quad \text{and} \quad \mathrm{x}_2,\ \mathrm{y}_2,\ \mathrm{z}_2 .$$

In the equations of motion of m_1, *i.e.* in the expression (14), we must use the coordinates and velocities of m_2 for the time t_2 defined by

$$c(t - t_2) = \Delta_2 = \sqrt{(\mathrm{x}_1 - x_2)^2 + (\mathrm{y}_1 - y_2)^2 + (\mathrm{z}_1 - z_2)^2} ,$$

and we have

$$\boldsymbol{x} = \mathrm{x}_1 - \mathrm{x}_2 = \mathrm{x}_1 - x_2 - \Delta_2 \xi_2 - \delta x_2 .$$

In (15) we must use the coordinates and velocities of m_1 for the time t_1 defined by

$$c(t - t_1) = \Delta_1 = \sqrt{(\mathrm{x}_2 - x_1)^2 + (\mathrm{y}_2 - y_1)^2 + (\mathrm{z}_2 - z_1)^2} ,$$

and we have

$$\boldsymbol{x} = x_1 - \mathrm{x}_2 + \Delta_1 \xi_1 + \delta x_1 .$$

Further, we have for use in (14)—

$$\mathrm{B}_1^{\,2}(1 - \phi_2^{\,2}) = \boldsymbol{r}^2(1 - \phi_2^{\,2} + \epsilon_2^{\,2}) + 2\boldsymbol{r}\chi_2 ,$$
$$\mathrm{A}_1\sqrt{1 - \phi_1^{\,2}} = -\Delta_2(1 - \xi_1\xi_2 - \eta_1\eta_2 - \zeta_1\zeta_2) + \boldsymbol{r}\epsilon_1 ,$$

and in (15)—

$$\mathrm{A}_2^{\,2}(1 - \phi_1^{\,2}) = \boldsymbol{r}^2(1 - \phi_1^{\,2} + \epsilon_1^{\,2}) - 2\boldsymbol{r}\chi_1 ,$$
$$\mathrm{B}_1\sqrt{1 - \phi_2^{\,2}} = -\Delta_1(1 - \xi_1\xi_2 - \eta_1\eta_2 - \zeta_1\zeta_2) - \boldsymbol{r}\epsilon_2 .$$

The expression for C is the same in both cases.

We find then,

(I.)
$$(\mathrm{X})_1 = \frac{k^2 m_1 m_2}{\mathrm{B}_1^{\,3}} \left\{ -\boldsymbol{x} - \delta x_2 - \frac{\boldsymbol{r}\epsilon_1\xi_2}{1 - \xi_1\xi_2 - \eta_1\eta_2 - \zeta_1\zeta_2} \right\},$$
$$(\mathrm{X})_2 = \frac{k^2 m_1 m_2}{\mathrm{A}_2^{\,3}} \left\{ \boldsymbol{x} - \delta x_1 + \frac{\boldsymbol{r}\epsilon_2\xi_1}{1 - \xi_1\xi_2 - \eta_1\eta_2 - \zeta_1\zeta_2} \right\}. \tag{16}$$

This form of the equations is not unique. We can multiply by any power of C, or make more complicated alterations, for which the reader is referred to Poincaré.

Multiplying by C we get—

$$(\mathrm{X})_1 = \frac{k^2 m_1 m_2}{\mathrm{B}_1^{\,3}\sqrt{(1 - \phi_1^{\,2})(1 - \phi_2^{\,2})}} \{ -(\boldsymbol{x} + \delta x_2)(1 - \xi_1\xi_2 - \eta_1\eta_2 - \zeta_1\zeta_2) - \boldsymbol{r}\epsilon_1\xi_2 \} . \tag{17}$$

(II.) $$(\mathrm{X})_2 = \frac{k^2 m_1 m_2}{\mathrm{A}_2{}^3 \sqrt{(1-\phi_1{}^2)(1-\phi_2{}^2)}} \{(\boldsymbol{x} - \delta x_1)(1 - \xi_1\xi_2 - \eta_1\eta_2 - \zeta_1\zeta_2) + \boldsymbol{r}\epsilon_2\xi_1\} . \quad (17)$$

This last form (II.) is preferred by Lorentz (*l.c.*, p. 1239), because the corresponding Newtonian force, $\mathrm{X}_i = (\mathrm{X})_i \sqrt{1-\phi_i{}^2}$, does not contain the velocity of m_i. (I.) is the law adopted by Minkowski, presumably because it gives the simplest result for a planet of infinitesimal mass.

As has already been remarked, the values of ξ_2, η_2, ζ_2 in $(\mathrm{X})_1$, and those of ξ_1, η_1, ζ_1 in $(\mathrm{X})_2$ differ from the values at the time t for which the simultaneous coordinates are taken. If we are content to neglect third orders, however, we can assume all velocities to correspond to the time t. If the motion were quasi-stationary, *i.e.* if the accelerations could be neglected, the velocities would be constant, and also δx_2 and δx_1 would disappear. The equations (16) and (17) would in that case be rigorous.

7. We will first consider the law (I.). Introducing the developments of B_1 and A_2 we find—

$$(\mathrm{X})_1 = \frac{k^2 m_1 m_2}{\boldsymbol{r}^3} \left\{ -\boldsymbol{x} - \delta x_2 - \boldsymbol{r}\epsilon_1\xi_2 + \tfrac{3}{2}\boldsymbol{x}\epsilon_2{}^2 + 3\frac{\boldsymbol{x}}{\boldsymbol{r}}\chi_2 \right\} ,$$

$$(\mathrm{X})_2 = \frac{k^2 m_1 m_2}{\boldsymbol{r}^3} \left\{ \boldsymbol{x} - \delta x_1 + \boldsymbol{r}\epsilon_2\xi_1 - \tfrac{3}{2}\boldsymbol{x}\epsilon_1{}^2 + 3\frac{\boldsymbol{x}}{\boldsymbol{r}}\chi_1 \right\} \quad (18)$$

We thus have by (13)—

$$m_1 \frac{d^2 \mathrm{x}_1}{dt^2} = \frac{k^2 m_1 m_2}{\boldsymbol{r}^3} \left\{ -\boldsymbol{x} - \delta x_2 + 3\frac{\boldsymbol{x}}{\boldsymbol{r}}\chi_2 + \tfrac{3}{2}\boldsymbol{x}\epsilon_2{}^2 - \boldsymbol{r}\epsilon_1\xi_2 + \boldsymbol{r}\epsilon_1\xi_1 + \boldsymbol{x}\phi_1{}^2 \right\}$$

$$m_2 \frac{d^2 \mathrm{x}_2}{dt^2} = \frac{k^2 m_1 m_2}{\boldsymbol{r}^3} \left\{ \boldsymbol{x} - \delta x_1 + 3\frac{\boldsymbol{x}}{\boldsymbol{r}}\chi_1 - \tfrac{3}{2}\boldsymbol{x}\epsilon_1{}^2 + \boldsymbol{r}\epsilon_2\xi_1 - \boldsymbol{r}\epsilon_2\xi_2 - \boldsymbol{x}\phi_2{}^2 \right\} \quad (19)$$

Now we have, to second orders,—

$$\delta x_2 = \tfrac{1}{2}\Delta_2{}^2 \frac{d^2 x_2}{c^2 dt^2} = \tfrac{1}{2}\frac{k^2 m_1}{c^2} \cdot \frac{\boldsymbol{x}}{\boldsymbol{r}}$$

$$\delta x_1 = \tfrac{1}{2}\Delta_1{}^2 \frac{d^2 x_1}{c^2 dt^2} = -\tfrac{1}{2}\frac{k^2 m_2}{c^2} \cdot \frac{\boldsymbol{x}}{\boldsymbol{r}}$$

$$\chi_2 = \frac{\boldsymbol{x}}{\boldsymbol{r}}\delta x_2 + \frac{\boldsymbol{y}}{\boldsymbol{r}}\delta y_2 + \frac{\boldsymbol{z}}{\boldsymbol{r}}\delta z_2 = \tfrac{1}{2}\frac{k^2 m_1}{c^2}$$

$$\chi_1 = -\tfrac{1}{2}\frac{k^2 m_2}{c^2} .$$

Further, if we put

$$\mathrm{M} = m_1 + m_2 , \qquad m_1 = \mu\mathrm{M} , \qquad m_2 = (1-\mu)\mathrm{M} ,$$

$$\lambda^2 = \frac{k^2 \mathrm{M}}{c^2} ,$$

we find

$$\frac{d^2x_1}{dt^2} = \frac{k^2M}{r^3}(1-\mu)\left\{ -x + \mu\lambda^2\frac{x}{r} + (\xi_1-\xi_2)r\epsilon_1 + \tfrac{3}{2}x\epsilon_2^2 + x\phi_1^2 \right\},$$
$$\frac{d^2x_2}{dt^2} = \frac{k^2M}{r^3}\mu\left\{ x + (1-\mu)\lambda^2\frac{x}{r} + (\xi_1-\xi_2)r\epsilon_2 - \tfrac{3}{2}x\epsilon_1^2 - x\phi_2^2 \right\}. \qquad (20)$$

In these equations the system of reference is arbitrary. We can always, by a Lorentz-transformation, make the velocity of any point equal to zero. If there were a point in the system having no acceleration, like the centre of gravity in Newtonian mechanics, the equations could be simplified by taking that point as origin of the system of reference. Now the integrals of the motion of the centre of gravity do not exist in this simple form under the new law, but it is still possible to find a point without acceleration. The centre of gravity has the coordinates x_0, y_0, z_0, defined by—

$$x_0 = \mu x_1 + (1-\mu)x_2, \ldots$$

We have thus

$$\xi_0 = \mu\xi_1 + (1-\mu)\xi_2.$$

If we denote relative velocities by letters without index, thus—

$$\xi = \xi_1 - \xi_2, \qquad \phi^2 = \xi^2 + \eta^2 + \zeta^2,$$
$$r\epsilon = x\xi + y\eta + z\zeta, \qquad \text{etc.}$$

We have

$$\xi_1 = \xi_0 + (1-\mu)\xi, \qquad \xi_2 = \xi_0 - \mu\xi,$$
$$\epsilon_1 = \epsilon_0 + (1-\mu)\epsilon, \qquad \text{etc.}$$

We find at once from (20)—

$$\frac{d^2x_0}{dt^2} = \frac{k^2M\mu(1-\mu)}{r^3}\left\{ -(1-2\mu)\lambda^2\frac{x}{r} + r\xi(\epsilon_1+\epsilon_2) - \tfrac{3}{2}x(\epsilon_1^2-\epsilon_2^2) + x(\phi_1^2-\phi_2^2) \right\}. \qquad (21)$$

The right-hand member is of the second order, and is thus the same for the laws I. and II. In it we can, for the relative coordinates and velocities, use their values in ordinary Keplerian motion. If then we take the orbital plane for the plane of (x, y) and the axis of x through the perihelion, the equations (21) become—

$$\frac{dc\xi_0}{dt} = \mathrm{P} + \mathrm{Q}.c\xi_0 + \mathrm{R}.c\eta_0$$
$$\frac{dc\eta_0}{dt} = \mathrm{P}' + \mathrm{Q}'.c\xi_0 + \mathrm{R}'.c\eta_0$$
$$\frac{dc\zeta_0}{dt} = 0.$$

The coefficients P, Q, R, P′, Q′, R′ are periodic functions of t devoid of constant terms.* Consequently $c\xi_0$, $c\eta_0$ are also periodic, and $c\zeta_0$ is constant. The point the components of whose velocity are the non-periodic parts of $c\xi_0$, $c\eta_0$, $c\zeta_0$ therefore has no acceleration. If we perform a Lorentz-transformation to a new system having this point as its origin, then the mean values of ξ_0, η_0, and ζ_0 are zero. Thus, if we neglect the periodic terms (which are, moreover, of the second order, and would introduce into (20) only terms of the fourth order, and those multiplied by μ'), we have—

$$\xi_1 = (1-\mu)\xi, \quad \xi_2 = -\mu\xi, \quad \text{etc.}$$

By subtracting the second equation (20) from the first we find then—

$$\frac{d^2\boldsymbol{x}}{dt^2} = \frac{k^2\mathrm{M}}{\boldsymbol{r}^3}\{-\boldsymbol{x} + \boldsymbol{r}\epsilon\xi + \boldsymbol{x}\phi^2\} + \tag{22}$$
$$+ \frac{k^2\mathrm{M}}{\boldsymbol{r}^3}\mu'\left\{2\lambda^2\frac{\boldsymbol{x}}{\boldsymbol{r}} - 2\boldsymbol{r}\xi\epsilon - 3\boldsymbol{x}\phi^2 + \tfrac{3}{2}\mu''\boldsymbol{x}\epsilon^2\right\},$$

where

$$\mu' = \mu(1-\mu) = \frac{m_1 m_2}{\mathrm{M}^2}, \quad \mu'' = 1 - 2\mu = \frac{m_2 - m_1}{\mathrm{M}}.$$

If the mass of the planet is neglected we have $\mu' = 0$, and we find—

$$\frac{d^2\boldsymbol{x}}{dt^2} + k^2\mathrm{M}\frac{\boldsymbol{x}}{\boldsymbol{r}^3} = \frac{k^2\mathrm{M}}{\boldsymbol{r}^3}\{\boldsymbol{r}\epsilon\xi + \boldsymbol{x}\phi^2\}. \tag{23}$$

* If the true anomaly v is introduced as independent variable instead of t, I find—

$$\frac{dc\xi_0}{dv} = \mathrm{P}_1 + \mathrm{Q}_1 . c\xi_0 + \mathrm{R}_1 . c\eta_0, \quad \text{etc.}$$

$$\mathrm{P}_1 = \frac{k\sqrt{\mathrm{M}}}{p^{3/2}}\lambda^2\mu'\mu''\{e\cos 2v + e^2\cos v - \tfrac{3}{2}e^2\sin^2 v\cos v\},$$

$$\mathrm{P}'_1 = \frac{k\sqrt{\mathrm{M}}}{p^{3/2}}\lambda^2\mu'\mu''\{e\sin 2v + 2e^2\sin v - \tfrac{3}{2}e^2\sin^3 v\},$$

$$\mathrm{Q}_1 = -\frac{\lambda^2\mu'}{p}\{2\sin 2v - 3e\cos^2 v\sin v\},$$

$$\mathrm{R}_1 = \mathrm{Q}'_1 = \frac{\lambda^2\mu'}{p}\{2\cos 2v + 2e\cos v - 3e\sin^2 v\cos v\},$$

$$\mathrm{R}'_1 = \frac{\lambda^2\mu'}{p}\{2\sin 2v + 4e\sin v - 3e\sin^3 v\}.$$

These are not developments in powers of e, but are rigorous for all values of e. It may be mentioned that, if the first term within the brackets in (21), which arises from the non-stationary character of the motion, were neglected, P_1 would have a constant term, and we should find an acceleration of the centre of gravity.

Comparing (13) and (11) we find that (23) is equivalent to

$$\frac{d^2\boldsymbol{x}}{d\tau^2} + k^2 \mathrm{M}\frac{\boldsymbol{x}}{\boldsymbol{r}^3} = 0 . \qquad (24)$$

This is the equation given by Minkowski (*l.c.*, p. 110). It could, of course, have been derived by taking $\mu = 0$ from the beginning. In that case the acceleration of m_2 is zero, and we can take m_2 as the origin of our system of reference. Then $\xi_2 = \eta_2 = \zeta_2 = 0$, $\mathrm{B}_1 = \boldsymbol{r}$, $\mathrm{x}_1 = \boldsymbol{x}$, and (24) is derived at once from (14).

In (24) τ is the proper-time of the planet, or planeto-centric time; in (23) t is the time of the system of reference, which has been chosen with its origin in the mean position of the centre of gravity. Practically t will coincide with the proper-time of the Sun, or heliocentric time.

The equations for the other two coordinates are, of course, quite similar.

The equations for the law II. are derived from those for the law I. by multiplying the right-hand members by C. Introducing the relative velocities we find, neglecting fourth orders,

$$\mathrm{C} = \frac{1}{\sqrt{1 - \phi^2}} .$$

The only difference between the laws II. and I. is therefore that in (22) and (23) the term $\boldsymbol{x}\phi^2$ is changed to $\frac{1}{2}\boldsymbol{x}\phi^2$. The simple form (24) is no longer possible with the law II. We have, however, in accordance with (12), for this law—

$$\frac{d}{dt}\frac{d\boldsymbol{x}}{d\tau} + k^2\frac{\mathrm{M}}{\boldsymbol{r}^3} = 0 . \qquad (25)$$

The only previous investigations on the same subject which have come to my knowledge, beyond the quoted papers of Poincaré and Minkowski, are those by Mr. Wilkens ("Zur Elektronentheorie," *V.J.S.* 1904, p. 209) and by Mr. Wacker ("Ueber Gravitation und Elektromagnetismus," Inaugural Dissertation (Tübingen), 1909). Mr. Wilkens finds secular terms in all elements. Some of these have amounts which could not long remain undetected. Thus, *e.g.*, his secular perturbation of the mean distance corresponds, for the Earth, to a shortening of the year by 19 seconds per century. His formulæ, however, are not in conformity with the principle of relativity, and his results are not confirmed here.

Mr. Wacker gives the equations (25). His equations are thus in conformity with the principle of relativity, and he adopts the law II., and neglects the mass of the planet.

8. The equation (24) is the ordinary equation of Keplerian motion. Consequently, if we neglect the mass of the planet and

use planeto-centric time, the motion is Keplerian. Thus, *e.g.*, the integrals of areas are *

$$x\frac{dy}{d\tau} - y\frac{dx}{d\tau} = \mathrm{G}, \text{ etc.}$$

Or, introducing heliocentric time, to second orders —

$$x\frac{dy}{dt} - y\frac{dx}{dt} = \mathrm{G}\left\{1 - \tfrac{1}{2}\frac{\lambda^2}{p}(1 + e^2 + 2e\cos v)\right\}\dagger. \qquad (26)$$

In (22), on the other hand, it is not advantageous to introduce the proper-time of one of the bodies, since we should thereby lose the symmetry gained by the introduction of relative velocities and coordinates. In the second-order terms we can introduce the ordinary Keplerian motion. We find then, taking the orbital plane for plane of (x, y), for the two laws equally—

$$x\frac{dy}{dt} - y\frac{dx}{dt} = \text{const.} - (1 - 2\mu')\mathrm{G}_0\frac{\lambda^2}{p}e\cos v, \qquad (27)$$

where

$$\mathrm{G}_0 = a^2 n\sqrt{1 - e^2} = k\sqrt{\mathrm{M}}\sqrt{p},$$

and v is the true anomaly.

Similarly we find for the *vis-viva* integral from (24)—

$$\left(\frac{dx}{d\tau}\right)^2 + \left(\frac{dy}{d\tau}\right)^2 = k^2\mathrm{M}\left(\frac{2}{r} - \frac{1}{a}\right),$$

or in heliocentric time—

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{k^2\mathrm{M}\left(\frac{2}{r} - \frac{1}{a}\right)}{1 + \lambda^2\left(\frac{2}{r} - \frac{1}{a}\right)}. \qquad (28)$$

From (22), on the other hand, we find—

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 2\frac{k^2\mathrm{M}}{r} + \text{const.} - 4k^2\mathrm{M}\frac{\lambda^2}{a}\left(\frac{a}{r^2} - \frac{1}{r}\right) +$$
$$+ 2k^2\mathrm{M}\mu'\frac{\lambda^2}{a}\left[\frac{4a}{r^2} - \frac{5}{r} + \tfrac{1}{2}\mu''\left\{\frac{ap}{r^3} - 3\frac{a}{r^2} + \frac{3}{r}\right\}\right]. \qquad (29)$$

For the law II. we must add to (28) and (29) the term—

$$+ k^2\mathrm{M}\frac{\lambda^2}{a}\left(\frac{a}{r^2} - \frac{1}{r}\right) + \text{const.}$$

* As we shall now use only relative coordinates, the distinction between the different types of letters has become unnecessary, and is dropped.

† Mr. Wacker has a similar formula, *l.c.*, page 34.

The equation (28) can then also be written in the form given by Wacker (*l.c.*, page 55),

$$\mathrm{E} = \frac{k^2\mathrm{M}}{r} + \text{const.},$$

where

$$\mathrm{E} = \frac{1}{\sqrt{1-\phi^2}} - 1 = \tfrac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right](1 + \tfrac{3}{4}\phi^2 + \dots).$$

9. The equations (24) of course give, taking the orbital plane as plane of (x, y),—

$$\begin{aligned} x &= a(\cos u - e), \\ y &= a\sqrt{1-e^2}\sin u, \end{aligned} \qquad r = a(1 - e\cos u), \tag{30}$$

where

$$u - e\sin u = n\tau, \qquad a^3n^2 = k^2\mathrm{M}. \tag{31}$$

To introduce heliocentric time we have

$$\frac{dt}{d\tau} = \sqrt{1+(\phi)^2} = 1 + \tfrac{1}{2}\frac{\lambda^2}{a}\frac{1+e\cos u}{1-e\cos u},$$

$$d\tau = \frac{1-e\cos u}{n}du,$$

therefore

$$ndt = \left(1 + \tfrac{1}{2}\frac{\lambda^2}{a}\right)\left(1 - \frac{1-\tfrac{1}{2}\frac{\lambda^2}{a}}{1+\tfrac{1}{2}\frac{\lambda^2}{a}}e\cos u\right)du$$

or if we put

$$\begin{aligned} n' &= n\left(1 - \tfrac{1}{2}\frac{\lambda^2}{a}\right) = \frac{k^2\mathrm{M}}{a^{3/2}}\left(1 - \tfrac{1}{2}\frac{\lambda^2}{a}\right), \\ e' &= e\left(1 - \frac{\lambda^2}{a}\right), \end{aligned} \tag{32}$$

we have

$$u - e'\sin u = n't. \tag{33}$$

Consequently for the law I. the coordinates of a planet of infinitesimal mass are expressed by the ordinary formulæ (30) of elliptical motion, but to express the excentric anomaly in heliocentric time we must in the equation of Kepler (33) use a slightly different excentricity, and also Kepler's third law is no longer quite exact. The deviations from pure Keplerian motion are periodic and have λ^2 as a factor. Taking the semi-major axis of the Earth's orbit as unit of length, the value of λ^2 is

$$\lambda^2 = 9{\cdot}85 \times 10^{-9}.$$

All these periodic terms are thus entirely insensible. The length of the year appears to be different when measured in

geocentric or in heliocentric time. This, of course, is a purely formal change of unit, and has nothing to do with the number of days in the year.

The difference between heliocentric and geocentric time, apart from this change of unit, is purely periodic. We find

$$t = \tau\left(1 + \tfrac{1}{2}\frac{\lambda^2}{a}\right) + \frac{\lambda^2}{an} e \sin u\,, \qquad (34)$$

which is the formula given by Minkowski, *l.c.*, page 111. The value of the coefficient of sin u is, for the Earth, $0^s{\cdot}00083$.

10. For the integration of the equations (22), or the corresponding equations for the law II., the easiest way seems to be to use the Keplerian motion as a first approximation, and to compute "perturbations" by the method of variation of elements, introducing the radial, transversal, and orthogonal perturbing forces. Taking the orbital plane for plane of (x, y), these forces are found to be

$$\text{for law I.:}\quad \begin{aligned} S &= S_1 + S_2 + \mu'\{-3S_1 - (2 - \tfrac{3}{2}\mu'')S_2 + S_3\}\,,\\ T &= (1 - 2\mu')T_1\,,\\ W &= 0\,, \end{aligned}$$

$$\text{for law II.:}\quad \begin{aligned} S &= \tfrac{1}{2}S_1 + S_2 + \mu'\{-3S_1 - (2 - \tfrac{3}{2}\mu'')S_2 + S_3\}\,,\\ T &= (1 - 2\mu')T_1\,,\\ W &= 0\,, \end{aligned}$$

where

$$S_1 = k^2 M\frac{\phi^2}{r^2}\,, \qquad S_2 = k^2 M\frac{\epsilon^{\frac{1}{2}}}{r^2}\,, \qquad S_3 = 2\frac{\lambda^2}{r^3}\,,$$

$$T_1 = k^2 M\frac{x\eta - y\xi}{r^3}\epsilon\,.$$

In the perturbing forces we can use the undisturbed or Keplerian motion. We find, then, taking the axis of x towards the (unperturbed) perihelion,—

$$S_1 = \frac{\lambda^2}{r^3}(1 + e\cos u) = \frac{\lambda^2}{r^2}\cdot\frac{1 + e^2 + 2e\cos v}{p}\,,$$

$$S_2 = \frac{\lambda^2}{r^4}ae^2\sin^2 u = \frac{\lambda^2}{r^2}\cdot\frac{e^2\sin^2 v}{p}\,,$$

$$S_3 = \frac{\lambda^2}{r^4}ae\sqrt{1 - e^2}\sin u = \frac{\lambda^2}{r^3}e\sin v\,.$$

In the well-known equations for the variation of elements, we introduce as independent variable, instead of the time, either the excentric or the true anomaly, or the radius-vector, by—

$$dt = \frac{r}{an}du = \frac{r^2}{a^2n\sqrt{1 - e^2}}dv = \frac{\operatorname{cosec} v}{ane\sqrt{1 - e^2}}dr\,.$$

With the true anomaly as independent variable, the equations are then—

$$da = \frac{2a^2r^2}{p}\left[\mathrm{S}e\sin v + \mathrm{T}\frac{p}{r}\right]dv,$$

$$de = r^2\left[\mathrm{S}\sin v + \mathrm{T}(\cos u + \cos v)\right]dv,$$

$$ed\varpi = r^2\left[-\mathrm{S}\cos v + \mathrm{T}\left(1 + \frac{r}{p}\sin v\right)\right]dv,$$

$$d\epsilon_1 = -\frac{2r^3}{a\sqrt{1-e^2}}\mathrm{S}dv,$$

$$\delta\epsilon = \delta\epsilon_1 + \frac{e^2}{1+\sqrt{1-e^2}}\delta\varpi.$$

We find the following "perturbations":—

in δa:

from S_1 : $-2\lambda^2\frac{a^2}{r^2}e\cos u$,

„ S_2 : $\frac{2}{3}\lambda^2(1-e^2)\frac{a^3}{r^3} - 2\lambda^2\frac{a^2}{r^2}e\cos u$,

„ S_3 : $\frac{4\lambda^2}{(1-e^2)^2}(e\cos v + \frac{1}{4}e^2\cos 2v)$,

„ T_1 : $-\frac{2}{3}\lambda^2(1-e^2)\frac{a^3}{r^3}$;

in δe:

from $\mathrm{S}_1, \mathrm{S}_2, \mathrm{S}_3$: $\frac{1}{2}\frac{1-e^2}{ae}$ × the corresponding terms in δa,

„ T_1 : $\frac{1}{2}\frac{1-e^2}{ae}.\delta a + \frac{\lambda^2}{a}\cos v$;

in $e\delta\varpi$:

from S_1 : $-\frac{\lambda^2}{p}\left[(1+e^2)\sin v + \frac{1}{2}e\sin 2v + ev\right]$,

„ S_2 : $-\frac{\lambda^2}{p}\cdot\frac{1}{3}e^2\sin^3 v$,

„ S_3 : $-\frac{\lambda^2}{p}\left[2\sin v - \frac{1}{2}e\sin 2v - ev\right]$,

„ T_1 : $\frac{\lambda^2}{p}\left[-\frac{1}{2}e\sin 2v + \frac{1}{3}e^2\sin^3 v + ev\right]$;

in $\delta\epsilon_1$:

from S_1 : $2\frac{\lambda^2}{a}u - 4\frac{\lambda^2}{a\sqrt{1-e^2}}v$,

„ S_2 : $2\frac{\lambda^2}{a}u - 2\frac{\lambda^2}{a\sqrt{1-e^2}}(v - e\sin v)$,

„ S_3 : $-4\frac{\lambda^2}{a\sqrt{1-e^2}}v$.

(35)

The periodic terms, all of which have very short periods, can be entirely neglected. For the law I. we find that there are no secular terms not multiplied by μ', which is in accordance with what was found above. The terms which have μ' as a factor also give nothing secular in δa or δe. In $\delta\varpi$ we have—

$$\delta\varpi = 2\mu'\frac{\lambda^2}{p}v\,.$$

For the law II. we find this same term, but also a secular term not multiplied by μ, viz.—

$$\delta\varpi = \tfrac{1}{2}\frac{\lambda^2}{p}v\,. \qquad (36)$$

With the law II. we have thus a secular motion of the perihelia, even for a planet of infinitesimal mass.* The motion in one century is—

For Mercury† :	$\delta\varpi = +7''\cdot15$	$e\delta\varpi = +1''\cdot470$	
,, Venus :	1·43	0·009	
,, the Earth :	0·639	0·011	
,, Mars :	0·225	0·021	
,, Jupiter :	0·0104		
,, Saturn :	0·0023		(37)
,, Uranus :	0·00040		
,, Neptune :	0·00013		

For ☄ Encke : $\delta\varpi = +0''\cdot308$
,, ☄ Halley : 0·0071

The effect of the secular terms in $\delta\epsilon_1$ is, of course, only to make the observed mean motion disagree with the value derived from Kepler's third law. As we have no sufficiently accurate means of independently determining the mean distances, this can never be detected.

The formulæ (35) are valid for all excentricities, and can thus also be used for comets. The perturbations in $\delta\varpi$ remain of the same order as for planets. Those in δe and $\delta\epsilon_1$ become zero for a parabola. Only the perturbation in δa would appear to become very large at perihelion. We have, however,

$$\delta p = (1 - e^2)\delta a - 2ae\delta e\,,$$

from which we find, for both laws,

$$\delta p = -2\lambda^2(1 - 2\mu')e\cos v\,,$$

which does remain small for $e = 1$.

* This was also found by Mr. Wacker, who gives for Mercury 7″·2 per century (*l.c.*, page 58).

† The value 6″·69 quoted by Professor Lorentz in his Göttingen lecture (*l.c.*, page 1239) was a first approximation.

The two laws I. and II. are the only ones that have been actually proposed, but we can, without violating the principle of relativity, multiply the forces by any power of C, and consequently any (positive or negative or even fractional) multiple of the quantities (37) will be in agreement with that principle.

11. The final result by the method of the preceding article must, for the law I. and for $\mu = 0$, of course be the same as that derived in article 9, and contained in the formulæ (30), (32), (33). We have, since $\varpi_0 = 0$,—

$$x = a_0(\cos u_0 - e_0) + \delta x\,,$$
$$y = a_0\sqrt{1 - e_0^2}\sin u_0 + \delta y\,,$$
$$u_0 - e_0 \sin u_0 = n_0 t\,, \qquad a_0^3 n_0^2 = k^2 \mathrm{M}\,,$$

$$\delta x = \frac{x}{a}\delta a + \tfrac{3}{2}\frac{an}{r}\sin u \int \delta a dt - \frac{a^2}{p}\sin u\left[\delta\epsilon_1 + e\cos v(\delta\epsilon - \delta\varpi)\right] + \frac{ae^2}{\sqrt{1-e^2}}\sin u\delta\varpi - \left[\frac{a^2\sin^2 u}{r} + a\right]\delta e\,, \quad (38)$$

$$\delta y = \frac{y}{a}\delta a - \tfrac{3}{2}\frac{an\sqrt{1-e^2}\cos u}{r}\int \delta a dt + \frac{a^2}{p}\sqrt{1-e^2}\cos u\left[\delta\epsilon_1 + e\cos v(\delta\epsilon - \delta\varpi)\right] - ae\delta\varpi + \left[\frac{a^2\sqrt{1-e^2}}{r}\cos u - \frac{ae}{\sqrt{1-e^2}}\right]\sin u\delta e\,,$$

where δa, δe, $e\delta\varpi$, and $\delta\epsilon_1$ must be collected from (35), taking the values for $\mathrm{S}_1 + \mathrm{S}_2 + \mathrm{T}_1$, and

$$e(\delta\epsilon - \delta\varpi) = e\delta\epsilon_1 - \sqrt{1-e^2}.e\delta\varpi\,.$$

On the other hand, from article 9, we have—

$$\delta x = -a\sin u\Delta u + (\cos u - e)\Delta a - a\Delta e\,,$$
$$\delta y = a\sqrt{1-e^2}\cos u\Delta u + a\sqrt{1-e^2}\sin u\Delta a - \frac{ae}{\sqrt{1-e^2}}\sin u\Delta e\,, \quad (39)$$

where

$$r\Delta u = -(\tfrac{3}{2}\Delta a + \tfrac{1}{2}\lambda^2)(u - e\sin u) + (a\Delta e - \lambda^2 e)\sin u\,,$$

and we have put

$$a = a_0 + \Delta a\,, \qquad e = e_0 + \Delta e\,.$$

By equating the values of δx and δy given by (38) and (39) we must then find for Δa and Δe constant values. To the first power of e I find—

$$\Delta a = \lambda^2\,, \qquad \Delta e = \tfrac{3}{2}\frac{\lambda^2}{a}.e\,.$$

12. In the motion of the Moon we have first the same terms as in the planetary motion, but M in the expression for λ^2 is now the sum of the masses of the Earth and Moon. The motion of the perigee in one century for the law II. is found from (36)—

$$\delta\varpi = +0''.0101 .$$

But, in addition to this, the disturbing force of the Sun is changed. To find the effect of this, we must use the first equation of (20), first taking for m_1 the Moon and for m_2 the Sun, and then for m_1 the Earth and for m_2 the Sun. The difference of the two equations thus derived—without the first term, which gives the ordinary Newtonian perturbation—will furnish the new terms in the acceleration of the Moon with respect to the Earth.

We can neglect the masses of the Moon and the Earth, and the acceleration of the Sun. Then, if we take the origin of the system of reference in the Sun, we find for the additional terms—

$$\frac{d^2x}{dt^2} = k^2m\left\{\frac{\Delta\epsilon_1\xi_1}{\Delta^3} + \frac{x_1{\phi_1}^2}{\Delta^3} - \frac{\rho\epsilon_2\xi_2}{\rho^3} - \frac{x_2{\phi_2}^2}{\rho^3}\right\},$$

where the suffix 1 refers to the Moon and 2 to the Earth, and where

$$\Delta = \text{distance Sun} - \text{Moon},$$
$$\rho = \quad ,, \quad \text{Sun} - \text{Earth},$$

and m is the Sun's mass. For the law II. the terms involving ${\phi_1}^2$ and ${\phi_2}^2$ are halved.

A development in series cannot be avoided. We will retain only the terms of the first order in the parallax, and in the inclination and excentricities, and of these only those which can give secular perturbations, *i.e.* which do not contain the Sun's longitude. I find—

$$\frac{d^2x}{dt^2} = -\frac{k^2m\nu}{\rho}\left[\tfrac{1}{2}\left(\frac{\lambda'}{\sqrt{a}} - a\nu\right)\cos l + \tfrac{3}{4}a\nu e + \tfrac{1}{2}\left(\frac{\lambda'}{\sqrt{a}} - \tfrac{1}{2}a\nu\right)e\cos 2l\right]$$
$$+\frac{k^2m\nu}{\rho}\left[\left(\frac{\lambda'}{\sqrt{a}} - \tfrac{1}{2}a\nu\right)\cos l + \tfrac{3}{4}a\nu e + \left(\frac{\lambda'}{\sqrt{a}} - \tfrac{1}{4}a\nu\right)e\cos 2l\right],$$

$$\frac{d^2y}{dt^2} = -\frac{k^2m\nu}{\rho}\left[\tfrac{1}{2}\left(\frac{\lambda'}{\sqrt{a}} - a\nu\right)\sin l + \tfrac{1}{2}\left(\frac{\lambda'}{\sqrt{a}} - \tfrac{1}{2}a\nu\right)e\sin 2l\right]$$
$$+\frac{k^2m\nu}{\rho}\left[\left(\frac{\lambda'}{\sqrt{a}} - \tfrac{1}{2}a\nu\right)\sin l + \left(\frac{\lambda'}{\sqrt{a}} - \tfrac{1}{4}a\nu\right)e\sin 2l\right],$$

$$\frac{d^2z}{dt^2} = \tfrac{1}{2}\frac{k^2m\nu}{\rho}\left(\frac{\lambda'}{\sqrt{a}} - a\nu\right)\sin i\sin(l - \Omega)$$
$$-\frac{k^2m\nu}{\rho}\left(\frac{\lambda'}{\sqrt{a}} - \tfrac{3}{2}a\nu\right)\sin i\sin(l - \Omega).$$

The second lines must be halved for law II. We have put—

$$\lambda' = \frac{k\sqrt{\mathrm{M}}}{c}, \qquad \mathrm{M} = \text{mass of Earth + Moon},$$

$\nu = n'/c$, $n' =$ mean motion of Sun.

$l =$ the Moon's mean anomaly.

I find then for the two laws—

	I.	II.
S	$\frac{1}{2}\frac{m}{\mathrm{M}}\frac{\lambda'\nu}{\rho\sqrt{a}}(1+e\cos l)$,	$\frac{1}{4}\frac{m}{\mathrm{M}}\frac{a\nu^2}{\rho}(1-e\cos l)$,
T	$-\frac{1}{2}\frac{m}{\mathrm{M}}\frac{\lambda'\nu}{\rho\sqrt{a}}e\sin l$,	0,
W	$-\frac{1}{2}\frac{m}{\mathrm{M}}\frac{\nu}{\rho}\left(\frac{\lambda'}{\sqrt{a}}-2a\nu\right)i\sin(l-\Omega)$	$\frac{1}{4}\frac{m}{\mathrm{M}}\frac{a\nu^2}{\rho}i\sin(l-\Omega)$.

There are no secular perturbations in a, e, and i. In the other elements we find—

	I.	II.
$\delta\varpi$	$-\frac{m}{\mathrm{M}}\frac{a^{3/2}\lambda'\nu}{\rho}.nt$,	$-\frac{1}{4}\frac{m}{\mathrm{M}}\frac{a^3\nu^2}{\rho}.nt$,
$\delta\epsilon_1$	$-\frac{m}{\mathrm{M}}\frac{a^{3/2}\lambda'\nu}{\rho}.nt$,	$-\frac{1}{2}\frac{m}{\mathrm{M}}\frac{a^3\nu^2}{\rho}.nt$,
$\delta\Omega$	$-\frac{1}{4}\frac{m}{\mathrm{M}}\frac{a^2\nu}{\rho}\left(\frac{\lambda'}{\sqrt{a}}-2a\nu\right).nt$,	$+\frac{1}{8}\frac{m}{\mathrm{M}}\frac{a^3\nu^2}{\rho}.nt$.

The numerical values for one century are—

	I.	II.
$\delta\varpi$	$-1''\cdot272$	$-0''\cdot024$
$\delta\Omega$	$-0\cdot270$	$+0\cdot011$

These quantities are too small to be detected from observation. Even the value of $\delta\varpi$ for law I. is well within the limits of uncertainty of the observed value.

13. We are thus left with the motion of the perihelion of Mercury as the only effect which reaches an appreciable amount. Unfortunately this same motion presents the well-known excess of observation on theory, which has been explained by Seeliger by the attraction of the masses forming the zodiacal light.* Until

* "Das Zodiakallicht und die empirischen Glieder in der Bewegung der innern Planeten," *Münchener Sitzungsberichte*, 1906, p. 595.

we have some independent means of accurately determining this mass —which seems a very remote possibility indeed—any motion of the perihelion of Mercury within reasonable limits can be so explained. It will be remarked that the value (37) of $e\delta\varpi$ for Mercury is only little larger than the part of the motion (1″·203), which Seeliger explains by a rotation of the astronomical system of coordinates with respect to the "inertial system," *i.e.* the system to which the equations of motion are referred. If this rotation were rejected, and the term (37) were added, most of Seeliger's residuals would be very little changed. There would then be appreciable residuals only for the nodes of Mercury and Venus, and the perihelion of Mars. The node of Venus could be put right by increasing the density of Seeliger's second ellipsoid to about three times the value adopted by Seeliger (which would still leave it less than 1/2000 of that of the inner ellipsoid). Seeliger's rotation is practically determined from the motions of the node of Mercury and the perihelion of Mars, the excesses of which over their theoretical values, as determined by Newcomb, are 1·2 and 2·1 times their respective probable errors. If the density of the second ellipsoid were so determined as to represent the node of Venus exactly, the residual in the perihelion of Mars would be reduced to 1·6 times its p.e. The other residuals would not be much affected,* and the representation would thus be on the whole very satisfactory.

14. We now come to the effect of a Lorentz-transformation on the elements of the orbit. First consider the orbital plane. In the system (x, y, z, t), of which the origin is in the Sun, or in the mean centre of gravity if the mass of the planet is not negligible, this plane is—

$$ax + by + cz = 0 .$$

We must transform to simultaneous coordinates by (6) or (7). Since $x_2 = y_2 = z_2 = 0$ and $\xi_2 = \eta_2 = \zeta_2 = 0$ we have $\Delta_q = r_q$, $x_1 - x_2 = x$, etc. The transformed plane is thus—

$$a(x'_1 - x'_2) + b(y'_1 - y'_2) + c(z'_1 - z'_2) + \frac{q_1}{1 - q_1} r'_q(a\alpha + b\beta + c\gamma) = 0 ,$$

or

$$a'(x'_1 - x'_2) + b'(y'_1 - y'_2) + c'(z'_1 - z'_2) = 0 ,$$

where

$$\begin{aligned} a' &= a(1 - q_1) + q_1 \mathrm{C}\alpha , \\ b' &= b(1 - q_1) + q_1 \mathrm{C}\beta , \\ c' &= c(1 - q_1) + q_1 \mathrm{C}\gamma , \end{aligned} \qquad \mathrm{C} = a\alpha + b\beta + c\gamma .$$

* The residual for the node of Mars, which is found by Seeliger about equal to the probable error, would almost disappear.

The orbit thus remains fixed and plane after the transformation. Since

$$a = \sin i \sin \Omega ,$$
$$b = -\sin i \cos \Omega ,$$
$$c = \cos i ,$$

we find easily

$$\sin i d\Omega = q_1 C(\alpha \cos \Omega + \beta \sin \Omega)$$
$$di = q_1 C[(\alpha \sin \Omega - \beta \cos \Omega)\cos i - \gamma \sin i] \qquad (40)$$

If L and B are the longitude and latitude of the positive half of the axis of the transformation we have—

$$\alpha = \cos L \cos B , \qquad \beta = \sin L \cos B , \qquad \gamma = \sin B ,$$
$$C = \sin i \cos B \sin (\Omega - L) + \cos i \sin B ,$$
$$\alpha \cos \Omega + \beta \sin \Omega = \cos B \cos (\Omega - L) ,$$
$$\alpha \sin \Omega - \beta \cos \Omega = \cos B \sin (\Omega - L) .$$

If the plane to be transformed is the plane of (x, y) itself, we have in the system (x, y, z, t) $i_0 = 0$. The transformed position of the plane is then defined by

$$\tan i'_0 = \frac{q_1 \gamma \sqrt{\alpha^2 + \beta^2}}{1 - q_1(1 - \gamma^2)} , \qquad \tan \Omega'_0 = \frac{\alpha}{-\beta} ,$$

or

$$\tan i'_0 = \tfrac{1}{2} q_1 \frac{\sin 2B}{1 - q_1 \cos^2 B} , \qquad \Omega'_0 = L + 90^\circ . \qquad (41)$$

Let the plane of (x, y) be the ecliptic, and consider another plane of which the inclination and node in the system (x, y, z, t) are i and Ω. In the system (x', y', z', t') its inclination and node on the plane of (x', y') are $i + di$ and $\Omega + d\Omega$, as given by the formulæ (40). Let its inclination on the transformed ecliptic be $i + \Delta i$, $\Omega + \Delta\Omega$. Taking unity for the denominator of $\tan i'_0$ in (41) we find easily—

$$\sin i \Delta\Omega = \sin i d\Omega - q_1 \gamma \cos i(\alpha \cos \Omega + \beta \sin \Omega) ,$$
$$\Delta i = di \qquad + q_1 \gamma \cos i(\beta \cos \Omega - \alpha \sin \Omega) ,$$

or

$$\Delta\Omega = q_1 \cos^2 B \sin (\Omega - L) \cos (\Omega - L) ,$$
$$\Delta i = q_1 \sin i[\cos^2 B \sin^2 (\Omega - L) - \sin^2 B \cos i]$$
$$- q_1 \sin^2 i \sin B \cos B \sin (\Omega - L).$$

If the transformed plane be the equator, we have—

$$\Omega = \Upsilon = pt , \qquad i = \epsilon ,$$

where p is the constant of general precession in longitude. We have thus—

$$\Delta\Pi = -\tfrac{1}{2}q_1 \cos^2 B \sin 2L + q_1 \cos^2 B \cos 2L.pt.$$

If we take for the axis of the transformation the direction of the Sun's motion relatively to the fixed stars, and for q the velocity of this motion, divided by the velocity of light, so that in the system (x', y', z', t') the mean velocity of the stars is zero, we have—

$$L = 270^\circ, \qquad B = +55^\circ, \qquad q = 0{\cdot}000070.$$

We find—

$$\Delta\Pi = -0''{\cdot}0000044 \text{ per century.} \tag{42}$$

This is, on the basis of the principle of relativity, the theoretical difference between the constant of precession as determined from the fixed stars (system x', y', z', t') and from the motions in the solar system (system x, y, z, t).

15. If we take the orbital plane in the system (x, y, z, t) as the plane of (x, y), and choose the axis of x perpendicular to the axis of the transformation, we have—

$$a = 0, \qquad r_q = \beta y,$$

and the formulæ for transformation to simultaneous relative coordinates in the system (x', y', z', t') become—

$$x' = x, \qquad y' = y(1 - \beta^2 q_1), \qquad z' = -\beta\gamma q_1 y, \qquad ct' = \frac{ct - \beta q y}{\sqrt{1 - q^2}}.$$

The problem is now, if x, y are given as functions of t, to find the expression of x', y', z' as functions of t'. Knowing that the orbit remains plane, we can eliminate z' by combining with our Lorentz-transformation a rotation round the axis of x' by an angle I given by—

$$K \sin I = -\gamma\beta q_1,$$
$$K \cos I = 1 - \beta^2 q_1,$$

from which

$$K^2 = 1 - 2q_1 + 2q_1^2 = 1 + 2q_1\sqrt{1 - q^2}.$$

The new plane of x'_1, y'_1 is thus the transformed orbital plane, and we have—

$$x'_1 = x' = x, \quad y'_1 = y' \sec I = Ky.$$

If, to take a simple example, the orbit in (x, y, z, t) were a circle—

$$x = a \cos nt, \qquad y = a \sin nt,$$

we find

$$n't' = nt - \frac{an\beta q}{c} \sin nt\,, \qquad n' = n\sqrt{1-q^2}\,,$$

or putting

$$\epsilon = \frac{an\beta q}{c}\,,$$

we have

$$\phi - \epsilon \sin \phi = n't'\,, \tag{43}$$

$$x'_1 = a\cos\phi\,, \qquad y'_1 = a\mathrm{K}\sin\phi\,.$$

If the orbit in (x, y, z, t) is a "perturbed" ellipse, the formulæ of transformation are more complicated, but the differences between the real motion and pure elliptic motion remain of the second order and periodic in terms of t', if they were so in terms of t.

16. One of the consequences of the principle of relativity is that it must be impossible by observations on bodies belonging to one and the same system to detect a motion of the whole system. Suppose in the system of reference (x, y, z, t) the Sun to be at rest, and a planet to describe a circle with uniform velocity. This, of course, is a dynamically possible state of motion under the new law as well as under the old. In the system (x', y', z', t') derived from (x, y, z, t) by a Lorentz-transformation with the axis (α, β, γ) and the modulus q, the Sun and the planet have a common velocity $-cq$ in the direction (α, β, γ), and the relative orbit is no longer circular, but is defined by (43). Let, again, the orbital plane be chosen as plane of (x, y) in the first system of reference, and let the axis of x be perpendicular to the axis of the transformation. The observer belonging to the system has no means of ascertaining the position of the plane of (x', y'), he can only observe the plane of the orbit, *i.e.* the plane of (x'_1, y'_1). To him the velocity of the Sun is thus in the direction $(\alpha', \beta', \gamma')$, where—

$$\alpha' = 0\,, \qquad \gamma' = \sqrt{1-\beta'^2}\,,$$

$$\beta' = \cos(\mathrm{A}+\mathrm{I})\,, \text{ if } \beta = \cos\mathrm{A}\,,$$

or

$$\beta'\mathrm{K} = \beta\sqrt{1-q^2}.$$

Let the observer be on the Sun, and let a signal be sent him from the planet every time when this latter crosses the axis of y.

In the system (x, y, z, t) the intervals between these crossings are equal, and also the times required by the signal to reach the observer are equal: he will observe signals at equal intervals.

In the system (x', y', z', t') the intervals between the crossings are unequal, and the aberration-times are unequal, and these two effects must cancel each other.

The times of crossing are—

$$\phi = \frac{\pi}{2}\,, \qquad \frac{3\pi}{2}\,, \qquad \frac{5\pi}{2}\,, \qquad \text{etc.}$$

or

$$n't' = \frac{\pi}{2} - \epsilon, \qquad \frac{3\pi}{2} + \epsilon, \qquad \frac{5\pi}{2} - \epsilon, \qquad \text{etc.}$$

The intervals are therefore—

$$\frac{\pi + 2\epsilon}{n'}, \qquad \frac{\pi - 2\epsilon}{n'}, \qquad \text{etc.}$$

or

$$\frac{\pi}{n'} + \frac{2a\beta q}{c\sqrt{1-q^2}}, \qquad \frac{\pi}{n'} - \frac{2a\beta q}{c\sqrt{1-q^2}}, \qquad \text{etc.}$$

Let the aberration-time be called b/c. We find easily—

$$b^2 = (a\mathrm{K} \pm \beta' q b)^2 + q^2 b^2 (1 - \beta'^2),$$

the upper sign being taken for the crossings through the positive axis of y (first and third epoch), and the lower sign for those in which y is negative (middle epoch). We find then—

$$b_2 - b_1 = b_2 - b_3 = -\frac{2\beta' a\mathrm{K} q}{1 - q^2} = -\frac{2a\beta q}{\sqrt{1-q^2}}.$$

The corrections to be applied to the true intervals to derive the observed intervals are—

$$\frac{b_2 - b_1}{c}, \qquad \text{and} \qquad \frac{b_3 - b_2}{c}.$$

The observed intervals are thus seen to be equal, as in the system (x, y, z, t).

Dr. C. V. Burton,* following an idea of Maxwell, has proposed to determine the velocity of the solar system "with respect to the aether" from observations of eclipses of Jupiter's satellites. Apart from the complications introduced by the excentricities and inclinations, and by the observer being on the Earth instead of on the Sun, his method is to determine βq from the observed inequality in the intervals. If the principle of relativity be true, the result must necessarily be *nil*. The method could thus be used to verify the principle of relativity.

Mathematically speaking, the dynamical aspect of the principle is not directly concerned in this verification. The observable effect would be *nil* whatever the motions of the observer and the signaller, which are subjected to the same Lorentz-transformation, may be, even if they were dynamically impossible, so long as the principle is true for the geometrical relations of time, space, and

* *Phil. Mag.*, 1910 March, page 417.

rays of light. But in practice the coordinates are always derived from the integration of equations of motion, and they could not be transformed by a Lorentz-transformation unless the right-hand members of these equations of motion satisfy the conditions imposed by the principle of relativity.

If we take for cq the Sun's velocity relative to the fixed stars, the effect to be expected, in case the principle of relativity were not true, would be equivalent to an inequality in the time of eclipse of $0^s{\cdot}10$ with a period of 12 years, corresponding to

$$0^\circ{\cdot}00023, \qquad 0^\circ{\cdot}00012, \qquad 0^\circ{\cdot}00006, \qquad 0^\circ{\cdot}00002,$$

in the longitudes of the four satellites. Our observations, as well as our knowledge of the principal constants of the theory, must become many times more accurate than they are now, before it will be possible to detect the existence of an inequality of this amount.

17. The velocity of the solar system relatively to the fixed stars, on the other hand, is the same in any system of reference. If in the system (x', y', z', t') in which the Sun has the velocity $-cq$, the star has no velocity ($\xi'_* = \eta'_* = \xi'_* = 0$), then, transforming back to the system (x, y, z, t), in which the Sun has no velocity, we find for the star the velocity $\xi_* = \alpha cq$, $\eta_* = \beta cq$, $\zeta_* = \gamma cq$ (neglecting terms of the second order).

If in the system (x', y', z', t') two groups of stars have any systematic motion relatively to each other, or all stars have a systematic motion relatively to the axes of reference, the same will be the case in the system (x, y, z, t), apart from small changes (of the second order) in the constants defining the direction of motion relatively to the ecliptic, which are, however, of no importance, as they are constant corrections to quantities whose values must be derived from observation. If no such systematic motion exists in (x', y', z', t'), neither does it exist in (x, y, z, t).

This brings us to the question of the astronomical system of coordinates. Let this be (x, y, z, t). The observations are referred to a system of axes x′ y′, z′, which is the system of the fundamental catalogue. This system is reduced to (x, y, z) by a rotation—

$$\begin{aligned} x &= x' \cos pt + y' \sin pt, \\ y &= y' \cos pt - x' \sin pt, \\ z &= z', \end{aligned}$$

where p is the constant of precession. This constant is so determined that the fixed stars shall have no systematic rotation with respect to the system (x, y, z), and the assumption is then made that (x, y, z) coincides with (x, y, z), the system to which the equations of motion are referred. Whether this is so or not can only be decided by comparison with the motions in the solar

system. The problem has been exhaustively treated by Anding (*Encyclopädie der Math. Wissensch.*, vi., 2, page 3) and Seeliger ("Ueber die sogenannte absolute Bewegung," *Münchener Sitzungsber.*, 1906, page 85) and others. Since the nodes are fixed under the new law, and also the perihelia are practically fixed, with the exception of Mercury (for which see art. 13), and the correction (42) to the constant of precession is insensible, the problem is not changed in aspect by the introduction of the principle of relativity, and we need not further refer to it here.

18. Astronomical time is defined by the rotation of the Earth, and is measured by the hour-angle of the axis of x. Such deviations from uniformity as are derived from the action of known causes—which are, however, negligible, and remain negligible under the new law—are allowed for, and the thus corrected rotation is supposed to be uniform. Now a motion which is in a system (x, y, z, t) described by the formulæ of uniform rotation, is no longer so described in another system (x', y', z', t'). The definition of astronomical time would thus be dependent on the system of reference.

But the axis of x cannot be observed. Time is determined by the observation of the hour-angles of stars. In the new system (x', y', z', t') the direction in which we see the stars is also changed, and a compensation is effected.

Determinations of time are made at intervals, and are used to regulate the clock. Equal intervals on the clock are then supposed to be equal intervals of "time." The construction of the clock depends on the laws of the simple pendulum, which swings under the action of gravitation. We would thus be led to identify astronomical time with the proper-time of the pendulum-bob. Instead of this we can, of course, take the proper-time of the transit-room, or of the centre of the Earth. In this time as variable the motion of the Earth, adopting the law I. and neglecting its mass, is strictly Keplerian. But if we wish to consider the whole solar system, it is better to introduce heliocentric (or "barocentric") time. The difference has been found to be insensible [equation (34)]. The whole question of the definition of astronomical time is thus only of academic interest. So far as the interpretation of observations is concerned, we can identify astronomical time with the variable t of any system of reference we wish to use.

On the Determination of Places of (1) *Reference Stars,* (2) *Fundamental Stars, by Photographic Methods.* By H. H. Turner, D.Sc., F.R.S., Savilian Professor.

1. Although measures of star images on photographic plates have been proved susceptible of the highest accuracy, the photographic method has as yet not been seriously applied in the department of Astronomy usually called meridian work. The transit-circle is undoubtedly an effective instrument; and though it has imperfections—even numerous imperfections—they have been revealed so gradually that there has been time to devise a remedy for each before the next defect was made manifest. But the nett result is to make a very complex structure out of what was once a simple instrument. The original idea was that the transit-circle should describe the meridian; but the simplicity of this notion disappears when we must make elaborate determinations of pivot errors at intervals, and observe collimation level and azimuth several times a day (since they have diurnal variations). Originally the declination observations were to be referred to the graduations of a carefully divided circle; but the care of the instrument-maker is of little use unless it be supplemented by a most laborious determination of the division errors, to which may be added investigations of flexure (both of the tube and of its axis, and probably also of the circle, though little attention has so far been paid to this), and of the R – D which probably represents some slow release from strain after each setting. Finally, there is personal equation and its derivatives, for which the machinery of transit micrometers, gauze screens, reversing prism eyepieces, and so on have been introduced. Truly the original simplicity of the instrument is under a cloud! And we have said nothing of the suggested reversals side for side and end for end.

2. To these inherent instrumental imperfections are to be added the inadequate check on atmospheric refraction which arises from the meridian method. It is further assumed that systematic errors in R.A. can be sufficiently eliminated by observing "12-hour groups" of stars, though the way in which daylight affects one end or the other of the group has not received much attention. The fact that even the best observations are affected by systematic errors has been demonstrated by the great investigation of Professor Boss.

3. And thirdly, a new setting of the instrument is usually made for each individual star, involving special computations which might suffice for a number of neighbouring stars if the work were better arranged. Transit-circle observation is so familiar that the great costliness of it has been forgotten.

4. Nor is it desirable to remember the costliness and other defects until we have a better alternative in full working order. It is imperative to retain our present methods in their integrity

until we are sure that we can abandon them with advantage; but none the less it behoves us, in view of the new tools which have been lately put into our hands, to experiment with a view to finding such an alternative.

5. And since the prospects of success are distinctly good, it is advisable to delay any unusually heavy piece of transit-circle observing until we see whether this success is assured. For instance, not very long ago a comprehensive survey was made of the northern stars, to about magnitude 9·5, under the auspices of the *Astronomische Gesellschaft*. The expenditure of astronomical energy in making this survey was vast, and is still fresh in our minds. Some of the work is even yet barely completed, though it was begun thirty or forty years ago. It is now proposed to repeat this work, and to extend it to the southern hemisphere, by the same expensive methods, without inquiry of any kind whether a more expeditious and economical process can be devised. Indeed, some of the repetition has already been carried out. The contention of the present paper is that this procedure is wrong. One or two great Observatories (Greenwich and the Cape, for instance) may find little difficulty in carrying out selected portions of the repetition by utilising the considerable resources which have rightly been put at their disposal. To observatories with only a fraction of these resources, the rest will be an almost intolerable burden, the weight of which will grow with the advance of knowledge of alternative methods.

6. It may be remarked incidentally that the proposals for repetition of the northern hemisphere zones (the southern have not been previously surveyed, and their case is different) are chiefly based on a misunderstanding. The places of the Catalogues of the *Astronomische Gesellschaft* have been used to find constants for the plates of the Astrographic Catalogue. The constants thus derived are erratic, and the errors have been *rightly* ascribed to the errors of the adopted star places. But the further assumption has been *wrongly* made, that those errors are chiefly due to proper motion. *Our Oxford experience shows clearly that the errors due to proper motion are remarkably few.* The faint stars are, as a rule, moving slowly, and it is quite the exception when any sensible allowance has to be made. All the "reference stars" have been ledgered: we can turn over the pages of the ledgers and compare at a glance the positions of stars on plates taken several years apart: our longest interval is nearly 18 years now, and we have many intervals of 10. If the errors were chiefly due to proper motion we should see them in turning over the pages. The fact is, they are much the same in 1900 as they were in 1885 when the stars were observed at Cambridge; but there was a large "magnitude equation" in the Cambridge R.A.'s, which upsets the constants for a plate on which there are very few reference stars. The declinations are good, and give good plate constants. It is, in my opinion, very doubtful whether the re-observation of these stars with the Greenwich Transit-Circle at the present moment is worth the great expense incurred.

The Cambridge observations of 1885 (approx.), when the R.A.'s have been corrected for magnitude equation (determined from the Oxford photographic measures), will probably give as good plate constants as the new Greenwich places for 1910.*

7. But this is a digression from the main point, which is simply this:—Assuming that the places of at least 100,000 stars (the argument is stronger as the number increases) scattered over the 40,000 square degrees of the whole sky (say 10 per plate of 4 square degrees) are required, we should not set out to determine them with the transit-circle just yet.

8. A brief sketch of these views, and of one proposal for giving effect to them, was published in *The Observatory* for September 1910. The proposals have attracted attention, especially in America and Australia, and request has been made for a fuller statement. The present paper is submitted in response. It cannot be too clearly stated that there are two quite distinct questions:—

(1) How can the photographic method be applied to differential observations?

(2) How can it be applied to fundamental observations?

The distinction is indicated in the title of the paper by using the words (1) reference stars, and (2) fundamental stars. We shall consider first the differential observations.

I. Differential Observations of Reference Stars.

9. Before giving a general method applicable in all cases, let us take a simpler method applicable to zones near the equator but not to stars near the pole.

(*a*) Point a photographic telescope to a cœlostat and fix it firmly.

(*b*) Give an exposure to a region in R.A. α from time t_1 to t_2.

(*c*) Set back the cœlostat to the position it occupied at time t_1. Precisely 10^{min} later start it again and expose from $t_1 + 10^{min}$ to $t_2 + 10^{min}$. The stars of R.A. $\alpha + 10^{min}$ and of the same declination as before will be photographed on the same plate.

(*d*) Set back the cœlostat again and expose from $t_1 + 20^{min}$ to $t_2 + 20^{min}$, thus getting the region $\alpha + 20^{min}$.

(*e*) Continue the process as often as convenient. The interval 10^{min} is of course purely arbitrary, and may be varied at will. The point is to get on the same plate regions of precisely the same declination, and differing in R.A. by exactly known differences.

10. The process of measurement and reduction of such a plate is tolerably obvious. The standard coordinates of the first region (*a*) are computed with a plate centre α_0, say; those of the next with plate centre $\alpha_0 + 10^{min}$; of the next, with $\alpha_0 + 20^{min}$, and so on;

[* *Note added March* 23. Since these words were in type, an examination of the results of applying corrections for magnitude equation has shown that they do not go far enough. The remaining discrepancies are, however, due to accidental errors; certainly not to proper motion, which is the main point.]

and then the stars can be treated as all belonging to the same plate. We shall by accumulation get a number of fundamental stars on the plate, from which the plate constants can be deduced, and then we get the places of the "reference stars" (by differential methods).

11. In the above brief statement several points have been slurred over.

Firstly, how is the cœlostat mirror to be set back to the same precise position? Let a fixed point of light ("artificial star") be reflected by the cœlostat mirror into a fixed telescope fitted with a transit reticle. As the cœlostat is moved by the clock, the artificial star will transit the field. At one of the outside wires, and before exposure is commenced in the photographic telescope, the observer can, by means of the slow motion of the cœlostat, bring the star very nearly into the same position at the specified moment. He finds the error of his setting by observing the transit. There is no question of magnitude equation, since it is the same star every time. And of course it is easy to devise a more automatic method of registering the position of the cœlostat.

12. *Refraction.*—Secondly, there will be certain corrections required to the multiple plate which do not arise with simple plates. The refraction will not be the same throughout. The effect of refraction has been given in several papers (*e.g.*, *Mon. Not.*, lxxi. p. 108) as

$$\Delta x/\beta = \mathrm{X} - x(1+\mathrm{X}^2) - y\mathrm{XY} + \text{terms of 2nd order, etc.}$$
$$\Delta y/\beta = \mathrm{Y} - x\mathrm{XY} - y(1+\mathrm{Y}^2) + \text{etc.}$$

where (x, y) is the position of the star on the plate, (X, Y) are the coordinates of the zenith, and β is the constant of refraction. Now, during the exposure over several hours β will vary; but, from the nature of the conditions imposed, (X, Y) will remain the same. The changes in β will be determined from readings of the barometer and thermometer in the ordinary way. It may be noted that the terms of zero order in (x, y), which have not usually appeared in the calculations hitherto, do not even now appear in their entirety. We need only apply the variations of βX and βY due to the changes in β. The remainders are absorbed in the arbitrary position of the plate centre.

13. *Refraction near the horizon.*—We may have to deal with cases in which the law β tan Z.D., on which the above formulæ are based, no longer applies accurately. In this case X and Y will be modified, but will remain constants. There need be no difficulty in working out such special cases when they arise.

14. *Precession, Aberration, etc.*—The corrections to the places of the stars can be expressed in the form

$$15\Delta\alpha \cos\delta = f' \cos\delta + g \sin(\mathrm{G}+\alpha)\sin\delta + h \sin(\mathrm{H}+\alpha)$$
$$\Delta\delta \quad = i \cos\delta + g \cos(\mathrm{G}+\alpha) \quad + h \cos(\mathrm{H}+\alpha)\sin\delta.$$

It will suffice to consider one of these expressions—say, the second. If we have calculated it in any way for one plate centre α_0, what will be its values for another plate centre $(\alpha_0 + \theta)$? It will become

$$\Delta\delta = i \cos \delta + \{g \cos (G + \alpha) + h \cos (H + \alpha) \sin \delta\} \cos \theta - \{g \sin (G + \alpha) + h \sin (H + \alpha) \sin \delta\} \sin \theta.$$

15. Now we can divide the corrections into two parts, one part affecting the *plate centres* only, and the other part being the differential effect between any star at (x, y) and the centre $(0, 0)$. For the first part we have $\alpha = A$ and $\delta = D$: the term $i \cos D$ (with the similar term $f' \cos D$) is constant for all plates, and is absorbed in the unknown coordinates of the centre. We thus have to find the two constants

$$g \cos (G + A) + h \cos (H + A) \sin D$$
$$g \sin (G + A) + h \sin (H + A) \sin D$$

with two similar quantities in R.A., and having found them, we multiply by the appropriate values of $\cos \theta$ and $\sin \theta$ for each plate to get the corrections to plate centre for that particular plate.

16. Now consider the differential effects. Precession and nutation will cause a simple rotation of the field; aberration a change of scale. There is no distortion; and hence the corrections are of the form

$$\Delta x = ax + by \qquad \Delta y = -bx + ay.$$

To determine a and b it is sufficient to know Δx and Δy exactly for a single star, and the most convenient star to choose is one on the axis of y, with the same R.A. as the plate centre, but declination $D + \phi$, say. Hence $x = 0$ and $y = \tan \phi$, and $\Delta x = b \tan \phi$, $\Delta y = a \tan \phi$. Whence

$$b = f' \sin D + g \sin (G + \alpha) \cos D$$
$$a = i \sin D + h \cos (H + \alpha) \cos D$$

when the quantities f' g i h have been reduced to circular measure.

17. The constant factors $f' \sin D$ and $i \sin D$ can be neglected as absorbed in the unknown orientation and scale value of the multiple plate. The other terms can be expressed in the form $M \cos \theta + N \sin \theta$, where M and N are constants for the night.

We thus have to find six constants in all at the outset, and they must be multiplied by $\cos \theta$ or $\sin \theta$ for each region. A very simple computation of the form $ax + by + c$ then gives the correction to any individual star.

18. Let us consider a few figures. Suppose we take plates at 10-minute intervals over 3 hours—say, 20 plates. On these we should have perhaps 20×20 reference stars and $20 \times 1\frac{1}{2}$ "intermediaries" $= 30$. We must first correct these 30 stars before they can be used to determine the mean constants of the plate. But once these mean constants are determined, then we correct *them* for each region before deriving the standard coordinates from the measures. We thus, instead of applying corrections to 20 stars individually, correct three constants—a great reduction of labour. Anyone who has had experience of meridian work knows the labour of computing star corrections.

19. *Photographic registration of intervals.*—Before leaving this special case where a cœlostat is used, it may be pointed out that the details may be varied in many particulars. Thus the transits giving the angular R.A.'s θ of each plate may be observed photographically; they may even be registered on the same plate which records the star images. Thus let PT be the plate and OG the object-glass, let light from an artificial star at H be rendered

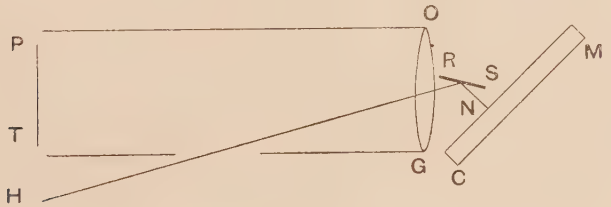


Fig. 1.

parallel by OG and then fall on a small reflector or prism RS rigidly fixed to the centre of the OG and at such an angle that the light is reflected normally to the cœlostat mirror CM. Then it will return on its path to H. But if the angle of the reflector RS be slightly altered the ray can be brought on to the plate PT, and as the cœlostat moves it will make a trail across the plate. This trail can be broken up into dots if the clock pendulum obscures the illumination of H at a regular part of its swing. (No electric contacts are necessary.) It remains to prevent the trails for the various plates interfering with each other. For this it is only necessary to make H part of a straight slit and uncover successive portions of it for the different plates. The *réseaux* which are now in general use will furnish the slits. There would be many advantages in having a large *réseau* fixed at PTH; an external part of it to be used for the star H, and the inside part to be impressed on the plate (which might be put film away from OG, and thus in contact with the *réseau*). The plate could then have the *réseau* impressed when it was in the telescope. There will be many practical details of this kind which experience will suggest.

20. *General case.*—But we must proceed to the general case. A cœlostat cannot be used for stars near the pole, or it may not be available in any case. The arrangement then takes another form, though the principles remain the same. These principles are as follows:—

(*a*) The star images must not trail, otherwise we shall not get the faint stars. Hence the plate must follow the star images. When we use a cœlostat these are fixed, and we can accordingly fix the plate; if we abandon the cœlostat, the plate must move with the star images.

(*b*) The motion of whatever part of the instrument moves must be registered, either on the plate itself or elsewhere. When we use a cœlostat, the mirror is the part that moves, and the motion is observed by means of an artificial star reflected from the mirror. If the plate moves, then a fixed artificial star (or preferably two stars, top and bottom) must register on the plate; and they must be obscured at regular intervals by the clock. There is no great difficulty in this. The chief difficulties suggest themselves in connection with (*a*).

21. *The Moving Plate. Theoretical.*—What *theoretical* errors are introduced by moving the plate in the focal plane of the objective? We defer for the moment the *practical* question of securing smooth and accurate motion, and assume that the plate can be moved accurately in its own plane either by translation or rotation, such motions being of course small. To fix the ideas, suppose we are to give an exposure of one minute on the equator sufficient to photograph 10th magnitude stars with an object-glass such as those in use for the astrographic work. The motion will be 30 sec. on either side of the centre, or say $7'{\cdot}5$, or in circular measure ·002.

22. It is worth while examining this matter with some care, since there are various misconceptions abroad. For instance, it was seriously objected by a critic (of great practical experience) that the star images, north and south of the centre, though they might be sufficiently good near the equator, would be unavoidably elongated *in declination* as we approach the pole. He was wrong in every particular; there is no trouble near the pole; if there is any trouble, it is at the equator, but it is far too small to be sensible.

23. Let us first take a few quite general considerations. By uniform translation and rotation of the plate in its own plane we can compensate terms of the first and second orders accurately, but this is all; we cannot deal with quantities of the third order (except possibly by varying the rate of these motions, which we put out of court). Hence we are liable to discrepancies of the third order. Now these arise from the motion, and vanish when the motion is zero. Also the motion is uniform to the first order, and there is no difficulty in compensating it accurately to that order near the middle line of the plate; hence the discrepancies must be of the second order in the motion, which is ·002 × ·002. The remaining factor to make up the third order will be one of the coordinates, and especially the declination coordinate. If we use a plate

covering 2° in declination (as for the Astrographic Chart) we are therefore liable to errors of 1° × ·002 × ·002 or 0″·015 at the edges of the plate, possibly reduced by a numerical factor. There should be no difficulty in calculating these with sufficient accuracy. Let us now consider them in detail.

24. Let PCE be the plane of the plate, C its centre, O the centre of the object-glass, so that OC is the focal length of the

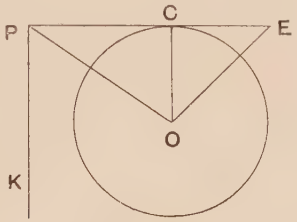


Fig. 2.

telescope. Let P and E be the projections, on the plane of the plate, of the pole and the equator respectively. Then the movement is a uniform rotation round OP, which can be resolved into a rotation round PC, and one round PK parallel to OC. The rotation round PK causes no theoretical difficulty. It can be completely compensated by a rotation of the plate in its own plane. At the pole this component exists alone, and there is no difficulty whatever.

25. But the rotation round PC is a rotation of the plate out of its plane, which is inadmissible. It is here that the motion

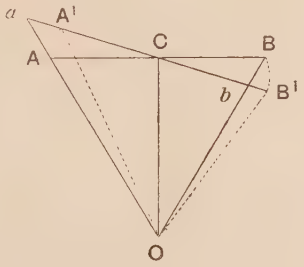


Fig. 3.

introduces inaccuracies, which are, as already remarked, of the second order in the motion. If we take a section of the plate perpendicular to PCE in fig. 2, this component will rotate the plate from the position AB to A′B′ (see fig. 3), so that the image of A, instead of falling at A′, will travel outwards to a, and that of B inwards from B′ to b. The errors introduced are in fact simply those due to tilt through the angle of rotation. The expressions for them are well known to be of the form

$$\Delta x = Xx^2 + Yxy \quad \text{and} \quad \Delta y = Xxy + Yy^2$$

where X and Y are the coordinates of the displaced normal. In the present case, if we take the direction of motion as axis of x, the tilt is ·002 and is wholly in X. Hence X = ·002 and Y = 0 and

$$\Delta x = \cdot 002x^2, \qquad \Delta y = \cdot 002xy.$$

Now if we make the exposures at the meridian of the plate, x itself is of the same order as the motion, and Δx is thus of the third order in the motion, *i.e.* 0″·002, and may be entirely neglected. There remains only the Δy term, which may amount to 0″·015 at the north and south edges. The nature of displacement due to tilt is shown in fig. 4, a rectangle ABDC being converted into the trapezium *abdc*. When exposures are confined to a narrow strip near the middle line QR, the width AB is so small that *ac* closes up to AC and *bd* to BD, so that the displacement in x is negligible. But there remains the travelling in declination, outwards from the centre. For any given star, y is constant and the displacement varies as x and is uniform. But the maximum

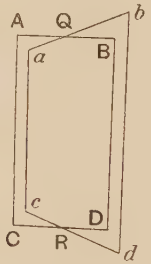


Fig. 4.

extent of the whole motion is 0″·03 for a plate 2° in width at the equator, and diminishes as $\cos^2 \delta$ (with the square of the linear motion), so that at 45° it is only one-half this, and at 60° only one-quarter.

26. In the case of the photographs of the planet Eros, much more serious movements than this were involved, and much trouble was anticipated therefrom. The event showed that the fears were groundless. For the present I shall assume that these much smaller motions may be entirely neglected; and I think this assumption might be made even if it were proposed to use much larger plates and much longer exposures.

27. *Movement of the Plate. Practical.*—We have arrived at the result that the movement to be given to the plate is a simple rotation in its own plane round P, the projection of the pole of the heavens (see fig. 5). Now OC is the focal length of the telescope and OPC = D the declination of the plate centre.

Thus

$$\mathrm{PC} = \mathrm{OC}.\ \cot \mathrm{D}.$$

We are accustomed to provide for the rotation of the telescope

in all directions; there should therefore be no new difficulty in providing for a slight rotation of an arm of about the length of the telescope (or less). In other words, for declinations greater than 45° (for which cot $\delta < 1$) the movement of the plate is simply and accurately arranged by mounting it on an arm which can turn about a pivot. The pivot must be movable in a long slot, so that the radius of rotation may be shortened as we approach the pole.

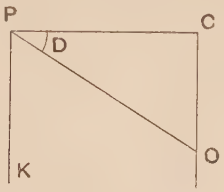


FIG. 5.

28. We have now to consider what to do when the arm PC becomes so long (as it does near the equator) that this arrangement is no longer convenient. Let us again remind ourselves of the size of the quantities involved. The total exposure being one minute of time, the angle round P is 15′. But this is *not* the angle described by the plate in its own plane; it is the angle of rotation round OP (fig. 5). The angle round PK is 15′ sin δ, which vanishes when we come to the equator, but is 15′ sin 45° or 10′ at dec. 45°. Any number of simple devices can be proposed to do the work with the required accuracy, and it is only a question of choosing the simplest and most convenient. A couple of examples will suffice.

29. *First Example.*—At the equator the movement is a pure translation, and can be provided by a straight slide. If this is made (and there seems no reason why it could not be made to work smoothly), all we want in addition for moderate declinations

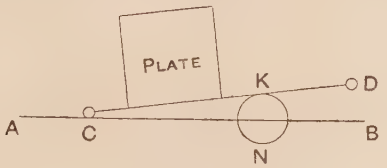


FIG. 6.

is a slight rotation. Let AB be the straight slide on which the plate-holder runs for equatorial stars, on two little wheels C and D perhaps. Introduce a button or wheel N at some fixed point of AB. Let the wheel C run on AB as before; as C approaches N the inclination of the edge CD to AB will increase slowly. The amount of rotation can be varied by altering either the height of the button or its position in AB.

30. *Second Example. Linkwork.*—The foregoing method depends on the use of slides. If it is preferred to use linkwork, which may be more easily constructed than a really straight slide, we know that the Peaucellier cell will give us rotations of any (small) curvature down to zero. If PEBFD be the cell (freely jointed at all these points), then PB × PD is constant. If B be joined to a fixed pivot A by a bar or a plate ABB which can turn freely about A, the points B will describe a circle about A. Thus the points D will describe the circle which is the inverse of this about P. Its centre will be in AP produced backwards, and its radius may be made infinite by choosing the position of the point A so that P lies on the circumference of the circle inverted; *i.e.* if AP = AB, which is therefore the correct position for equatorial motion. For north declinations A is nearer to P than this; for south, further away. There should be no difficulty in arranging this movement in one way or another. But it is to be remarked that when the declination of the plate centre is small, it is easy

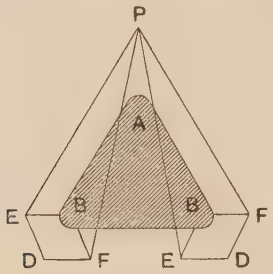


Fig. 7.

to use a cœlostat, so that this method is in reserve if the slides or linkwork prove troublesome. If the movement of the plate introduces errors, it is to be remarked that they will depend on the *relative* motion of the star and plate, which can be reversed in sign. Hence the errors are of the accidental and not of the systematic kind.

31. *General Method of Reduction.*—The general method of reduction will be tolerably clear from what precedes. The plate will contain a number of fundamental stars whose places are to be regarded as known. These places are to be corrected for differential star corrections and differential refraction, using the word "differential" in the sense above indicated. On comparing them with the measures we shall find differences Δx, Δy which are to be equated to

$$\Delta x = ax + by + c + kt, \qquad \Delta y = dx + ey + f + lt,$$

where a, b, c, d, e, f represent as usual the corrections for scale value, orientation, and centring; and kt, lt are terms due to

unknown clock-rate or settling of the instrument with the time t. These equations can be solved by least squares or an equivalent method. The coefficients of the unknown a, b, c and k are x, y, 1, and t, which are quite independent, so that the only further condition needful is a sufficiency of fundamental stars.

32. It may be remarked that it is not necessary, perhaps not even desirable, to have the fundamentals scattered among the different regions. A good plan would be to survey rather carefully by fundamental methods two particular regions, one near the beginning and one near the end. These would give independent determinations of

$$a,\ b,\ (c+kt_1) \quad \text{and} \quad a,\ b,\ (c+kt_2),$$

and this would afford a valuable check on the work.

33. There are other reasons why we shall probably find it desirable to modify our notion of fundamental points of reference. Hitherto these have been single stars; it will be more appropriate in future to make them *areas*. Thus the areas near the poles, the first points of Aries and Libra, and the points 90° distant from these, are of special importance. We shall avoid systematic errors by referring to the mean of a number of stars instead of to one only; and if a particular star turns out to be unsuitable, owing, for instance, to great or irregular proper motion, we can omit it without serious loss, and use the other stars near.

II. Fundamental Observations.

34. The transit-circle is not really a fundamental instrument, because its use is based on certain assumptions which are only approximately true. Thus it is assumed in Z.D. observations that the distance between two graduations of the circle remains the same in all positions, whereas the effect of gravity must vary. The resulting errors can be checked by turning the circle round on the instrument so that the gravitational effects are different; but if a horizontal circle can be used, the errors do not arise. In transit observations there is an even greater departure from a really fundamental check. The arcs ♈≏ and ≏♈ should add up to make 360°; but they are measured at opposite seasons of the year, or opposite periods of the day. We should compare arcs in the sky with fundamental subdivisions of 360°, measured on a horizontal circle or the equivalent, the measures of equality of the subdivisions being virtually simultaneous.

35. *The Fundamental Apparatus.*—Hence, as fundamental apparatus, I propose two horizontal telescopes at right angles, pointed to the same mirror.

To set them at right angles, a simple method is to have a glass cube,* silvered on all its faces, or at least on four consecutive faces.

* Of course a single vertical mirror, attached to a horizontal circle read by microscopes, would answer the same purpose, but the cube is simpler.

A spot of light in the focus E_1 can then be reflected from one face of the cube and refocussed in the focal plane; while another spot in the focal plane E_2 is returned to that plane from another face of the cube. If this is done in the four consecutive positions

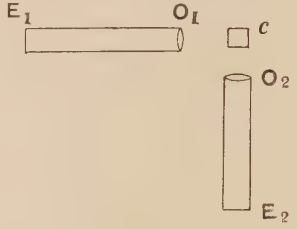


Fig. 8.

of the cube the angles between the telescopes (and incidentally those between the faces of the cube) can all be determined. The cube can then be turned upside down and the process repeated. If desired, telescopes could be collimated with the two shown, and used on the opposite faces of the cube, which would give additional checks.

36. The dimensions of the cube need only be a fraction of the diameter of the object-glasses if tests are made of the agreement of the collimation of the central portions of the O.G.'s with the whole. But of course a large cube is better than a small one. It should be carefully levelled, but the errors of level will only introduce errors of the second order into the angle between the telescopes. When the observations of adjustment have been made, the cube is to be removed (on a little railway, perhaps) and the mirror brought into position.

37. Now let T_1 and T_2 be the directions in which the telescopes point on the horizon (fig. 9), and let N be the normal to the mirror,

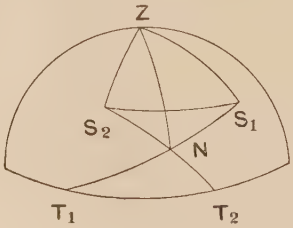


Fig. 9.

and S_1 S_2 the stars reflected in the mirror to the centre of the telescopes, so that $T_1N = NS_1$ and $T_2N = NS_2$. Hence $S_1S_2 = T_1T_2$ and is a right angle. We can thus measure right angles in the sky by use of this apparatus, affected of course by refraction. If the

azimuth of N falls midway between those of T_1 and T_2 the whole figure is symmetrical, and $ZS_1 = ZS_2$; but usually $ZS_1 \neq ZS_2$.

38. *The influence of refraction on the arc S_1S_2 is nearly constant.*—If we take refraction as k tan Z.D., which is approximately true to considerable zenith distances, then refraction will shorten the arc S_1S_2 by k tan ZS_1 cos ZS_1S_2 at one end, and by k tan ZS_2 cos ZS_2S_1 at the other. Now

$$\cos ZS_2 = \cos ZS_1 \cos S_1S_2 + \sin ZS_1 \sin S_1S_2 \cos ZS_1S_2,$$

and since $S_1S_2 = 90°$, this reduces to

$$\cos ZS_2 = \sin ZS_1 \cos ZS_1S_2,$$

hence the total influence of refraction on the arc S_1S_2 is

$$k\left(\frac{\cos ZS_1}{\cos ZS_2} + \frac{\cos ZS_2}{\cos ZS_1}\right)$$

and in the case when $ZS_1 = ZS_2$ this is $2k$. If ZS_1 differs from ZS_2, but not very seriously, we can put $\frac{\cos ZS_1}{\cos ZS_2} = 1 + n$, and then

$$\frac{\cos ZS_1}{\cos ZS_2} + \frac{\cos ZS_2}{\cos ZS_1} = 1 + n + (1+n)^{-1}$$
$$= 2 + n^2 \text{ to the 2nd order.}$$

Since $k = 57''$ about, when $n = 1/10$ $kn^2 = 0''{\cdot}6$, and when $n = 1/80$, $kn^2 < 0''{\cdot}01$. Hence the effect of refraction is sensibly constant when cos ZS_1 does not differ from cos ZS_2 by much more than 1 per cent.

Now if $ZS_2 = ZS_1 + x$,

$$\cos ZS_2 = \cos ZS_1 - x \sin ZS_1.$$
$$\therefore \quad n = x \tan ZS_1$$
$$x = n \cot ZS_1.$$

The minimum value of ZS_1 is 45°, and hence the maximum value of x is 1/80, or nearly 1°. When $ZS_1 = 70°$, cot $ZS_1 = 0{\cdot}36$ and

$$x = -0{\cdot}36 \qquad n = 15'.$$

Thus even at Z.D. 70° we may allow a motion of 15′ to a star or a field of stars without altering sensibly the effect of refraction on the angle S_1S_2.

39. *General Triangulation.*—Hence we have a simple method of laying out on the face of the sky arcs of 90°. Adopting an arbitrary centre for one plate, we find by an easy adjustment a

point on the other plate exactly 90° from it, and can, if we please, infer a point $90^\circ + 2k$ from it, where k is the coefficient of refraction at the particular moment. If a mirror placed within 15′ or more of the mid azimuth and at any convenient altitude will reflect two stars to these points, they are accurately 90° apart in the (apparent) heavens. There will seldom be two stars so accurately placed, but the fields surrounding the points can be connected by tolerably obvious methods with them. It only remains to select the points to be connected. We must remember aberration, which will alter the arcs (in the notation of § 48 below) by $k\{\sin(\alpha+\mathrm{T}+90^\circ)-\sin(\alpha+\mathrm{T})\}$ or $2k\sin 45^\circ\cos(\alpha+\mathrm{T}+45^\circ)$. The coefficient $1{\cdot}4k$ is thus stronger than either of those in § 48.

40. *Movement of the Plates.*—The points selected for comparison with an arc of 90° measured by terrestrial means will only occupy the theoretical position instantaneously. We must make the photographic exposures during (say) a minute of time. There is no difficulty in arranging for the plates to follow the stars on the plan already sketched, except that we have now two plates to move instead of one. For the actual pole of the heavens we now substitute the image of the pole as reflected in the mirror, and it is round this that the plates must be made to rotate.

41. *Movement of the Mirror no advantage.*—The question naturally arises whether any good purpose can be served by moving the mirror itself. I have given much thought to this point, and have concluded that the answer is in the negative. The following considerations sum up the reasoning briefly:—

(*a*) Any arc S_1S_2 in the sky remains unaltered in length. Hence the movement of S_1 resolved along the arc is always equal to that of S_2 similarly resolved. But the components perpendicular to the arc will generally be different.

(*b*) A movement of the mirror is also powerless to affect the length of S_1S_2. Hence the resolved motions along the arc are again equal. Choosing any arbitrary *direction* of motion for the mirror, we can therefore adjust its *rate* so as to compensate the motions of the stars along the arc.

(*c*) But we have still left two different components at right angles, and we have only one disposable constant (the direction of motion) to modify them. We can destroy one, but we should still have to move both plates. For if we destroy one component we only do so for the centre of the plate, which must still rotate.

42. To supplement this general reasoning the particular case which follows is examined in some detail by a method which can easily be extended to other cases. But the general result is the same. The mirror is better fixed, and we must move both plates.

43. *Equatorial Arcs.*—One set of measures suggests itself at once. Let the arcs be equatorial arcs so that $PS_1 = PS_2 = 90^\circ$. The arc S_1ES_2 is reflected to the position T_1AT_2 by the mirror. If N be the direction of the mirror normal, then since E and A are corresponding points, EN = NA, or the altitude of N is half the

colatitude. Let the latitude be ϕ. Then

$$\cos NT = \cos 45^\circ \cos \tfrac{1}{2}(90^\circ - \phi).$$

When

$\phi =$	0°	10°	20°	30°	40°	50°	60°
$NT =$	60	57	54	52	50	48	46
$ZS =$	45	46	49	52	57	63	69

To avoid large angles of incidence we must therefore make these observations at observatories not near the equator. We shall

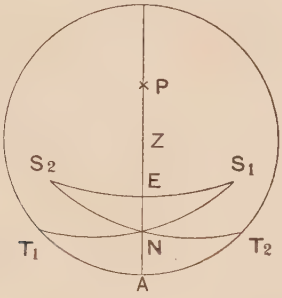


FIG. 10.

presently see that the appropriate work for tropical observatories is the observation of fundamental declinations.

44. *Possible Movement of Mirror for Equatorial Arcs.*—As above remarked, we will examine in detail the possible movement of the mirror for this case.

The movement of the stars S_1 and S_2 is wholly in the arc joining them. But the motion of N inevitably introduces com-

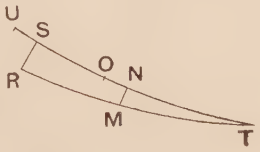


FIG. 11.

ponents at right angles to this arc. Let us consider first a movement of N along the arc TN to O (fig. 11). Then if S is carried to U we have

$$SU = 2NO.$$

Next consider a movement of N to M, where NM is perpendicular to TN. Then if S is moved to R we have

$$\begin{aligned} SR &= \angle T \times \sin ST = \angle T \sin 2TN \\ &= 2NM \cos TN. \end{aligned}$$

Now it is tolerably obvious that the best motion of N must be symmetrical; for everything else in fig. 10 is symmetrical. The only symmetrical motions are along PN and perpendicular to it. The first is clearly of no use, and the second gives displacements

$$\Delta \sin S_2NZ \text{ along } T_2N$$
$$\text{and } \Delta \cos S_2NZ \text{ at right angles to it.}$$

These produce in S_2 the motions

$$2\Delta \sin S_2NZ \text{ along } T_2S_2$$
$$\text{and } 2\Delta \cos T_2N \cos S_2NZ \text{ at right angles to it.}$$

We have now to resolve these along and perpendicular to S_2S_1. Clearly the angle NS_2S_1 is equal to NT_2A, and we shall simplify matters by expressing our angles as parts of the triangle NT_2A in the usual way; capital letters for the angles (omit the suffix), and small letters for the sides. Thus we denote the angles

$$ANT_2,\ NT_2A, \text{ and } NAT_2 \text{ by N, T, and } A = 90^\circ,$$
$$\text{and the sides } AT_2,\ NA, \text{ and } NT_2 \text{ by } N = 45^\circ,\ t \text{ and } a.$$

Then the motion of S_2 resolved along S_1S_2 is

$$2\Delta \sin N \cos T + 2\Delta \cos a \cos N \sin T,$$

and the motion of S_2 resolved at right angles is

$$2\Delta \sin N \sin T - 2\Delta \cos a \cos N \cos T$$
$$= 2\Delta \sin N \sin T (1 - \cos^2 a) = 2\Delta \sin n \sin t$$
$$= \Delta \left(\cos \frac{\lambda}{2} - \sin \frac{\lambda}{2}\right) = \tfrac{1}{2}\Delta \text{ approx. for latitude } 50^\circ.$$

45. Hence if we *do* move the mirror, there will still be considerable trailing. As before remarked, the trailing will be at right angles to the direction of measurement and may not matter; but we shall lose faint stars. It seems better to leave the mirror fixed and to slide the plates. At the equator the movement is a simple slide; and since the arc S_1S_2 is reflected into the horizontal arc T_1T_2, the movement is horizontal in each case, though one plate is at right angles to the other. It is desirable to have the motions accurately the same: there are many ways of attaining this result by mechanically connecting the plates.

46. *Possible Alternative Programme for Equatorial Arcs.*—It seems desirable, at this tentative stage, to draw attention to the following alternative plan. The disadvantage of it lies in the large angles of incidence that are necessary, but it is quite possible that the dangers which these large angles suggest are imaginary, or can be evaded; and if so, the method has many advantages. Let

there be *three* telescopes, AB, CD, EF, all pointed to a cœlostat mirror M, and all in the plane of the *equator* (fig. 12). Let AB and CD lie east and west, and be therefore also horizontal; and let the cœlostat run on rails, so that when it is out of the way the telescopes AB and CD can be collimated with one another. *This*

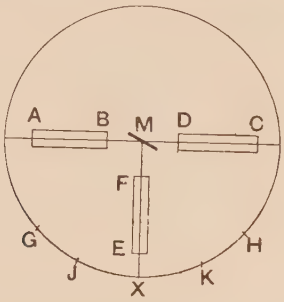


FIG. 12.

is the only fundamental adjustment. By it we lay out an angle of 180°, and we proceed to deduce two angles of 90° as follows:—

Let G and H be stars (or fields of stars) on the equator in hour angles $\pm$ 3^h, so that GH is approximately 90°. Set the cœlostat mirror normal to the direction J, midway between G and X. Then H will be reflected into the telescope BA, and G into the telescope FE. We thus compare the arc GH with the angle between AB and EF.

Next set the mirror normal to the direction K, midway between H and X. Then H will be reflected into FE and G into DC. We thus compare the same arc GH with the angle between EF and CD.

It is clear that we can deduce the length of GH and the angles between the telescopes.

The use of a cœlostat mirror avoids any need for moving plates. The only disadvantage is that we use angles of 67½° incidence, which is somewhat alarming. Is it really unsafe? The answer must be given by experiment.

47. *Secondary division.*—But a slight modification of this plan will be of great use in subdivision of the primary arcs. By the method already sketched we can measure four equatorial arcs of 90°, starting from an assumed Y, say YX, XY, YZ, ZY. We must now divide each of these into two, which can be done as follows. Set two telescopes (fig. 13) AB, CD *in the equator* and in hour angles 1½^h east and west, pointing to a cœlostat mirror M. When one of the arcs XY comes to the symmetrical position, the mirror M can be placed so as to reflect EX and EY into the telescopes alternately. These arcs of 45° can thus be made strictly equal; and since their sum is known to be 90°, their lengths are determined. In this case the angles of incidence have only half the previous values; they are 11° 15′ and 33° 45′ respectively.

The arcs are affected by refraction, but the effects can be made strictly symmetrical, though the details need not be here considered.

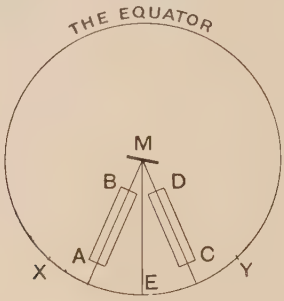


FIG. 13.

48. *Can the Secondary Method be used as primary?*—It might at first appear that in this way each of eight equatorial arcs could be compared with the next, with no apparatus beyond two telescopes and a cœlostat, and that this is in itself a primary subdivision of the circumference. [The number of arcs (eight) is rather large, but they can easily be reduced to six by inclining the telescopes at 60° instead of 45°.]

But we must remember aberration, which affects the arcs differently when the separate comparisons are made. For simplicity, take aberration as $k \sin(\alpha + T)$, where T represents the time of year. Then the effect on the difference of two adjacent arcs of length β is

$$k\{\sin(\alpha+T+\beta) - \sin(\alpha+T)\} - k\{\sin(\alpha+T) - \sin(\alpha+T-\beta)\}$$
$$= 2k(\cos\beta - 1)\sin(\alpha+T)$$
$$= -4k\sin^2\frac{\beta}{2}\sin(\alpha+T).$$

When $\beta = 45°$, $4\sin^2\frac{\beta}{2} = 0{\cdot}6$; and when $\beta = 60°$, $4\sin^2\frac{\beta}{2} = 1$.

Now we can compare any particular pair of arcs at times of year differing by a few months, and so we can vary $\sin(\alpha + T)$ to some extent; and in this way we may get a sufficiently strong determination of aberration which can be used for correcting the comparisons. The use of the primary subdivision was suggested to give us this determination as strong as it can be made, and then we can use it in subdividing the arcs of 90° as above sketched. But I do not feel sure at present that the secondary method may not suffice.

49. *Tertiary division.*—We can in any case get eight equal arcs of 45° all round the equator; and note that these are not represented by single stars but by *regions*, of a size depending on the field of good definition of the telescopes employed. When the

declinations of these regions have also been determined (as presently to be described), we can fill in the R.A. and Dec. of intermediate points on the equator by the differential method; for we have well-determined regions of stars at 3-hour intervals, which will give us constants of the plate, and clock-rate, and this is all we want to apply the differential method.

Having got the equator mapped out we can determine the R.A. (not the N.P.D.) of any region in the sky by measuring an arc of 90° from some point in the equator. For let (A, D) be the point on the sphere. We must measure its distance from the point $(A \pm 90°, 0°)$ on the equator, and we must do so at a time when when these are at equal zenith distances. Let Z be the hour angle of the first point; then its Z.D. in latitude ϕ is given by

$$\cos \text{Z.D.} = \sin \phi \sin \text{D} + \cos \phi \cos \text{D} \sin \text{Z},$$

and that of the second by

$$\cos \text{Z.D.} = \pm \cos \phi \cos \text{Z}.$$

Thus we have, taking the upper sign for simplicity,

$$\cos \text{Z} - \cos \text{D} \sin \text{Z} = \sin \text{D} \tan \phi,$$

or if $\cos \text{D} = \tan \theta$, so that $\theta < 45°$,

$$\cos (\text{Z} + \theta) = \cos \theta \sin \text{D} \tan \phi.$$

The solution is impossible if

$$\cos \theta \sin \text{D} \tan \phi > 1,$$
$$i.e. \text{ if } \tan^2 \phi > 2 \operatorname{cosec}^2 \text{D} - 1.$$

But the disabilities arise chiefly in high latitudes and need not trouble us.

50. We can thus determine the R.A. of any selected region on the sky. The selection of special regions is a totally different matter that need not concern us here; it suffices to show that the R.A. of any selected region *can* be found.

51. *Fundamental Polar Distances.*—The next step is to measure the distance from the poles to a number of points on the equator. In latitudes $> 45°$ a point on the equator is never at the same altitude as the pole; in latitude 45° it is so once only in 24 hours, in lower latitudes twice. There will be advantages in making the observations in low latitudes, so as to compare results east and west. There will also be the comparison of observations in the north and south hemispheres, so that we should get a strong fundamental determination of zero declination.

One particular latitude is worthy of remark. If the ends of an equatorial arc of 90° are at the same altitude as the Pole, then the determination of refraction from the equatorial arcs is directly available for the N.P.D. observations. The latitude is $\tan^{-1}(\cos 45^\circ)$ or $35^\circ\ 15'$ (Flagstaff, Tokio, La Plata, Cape, Adelaide, Mt. Wilson, Sydney, etc. etc.).

52. The details require only brief mention. The azimuth of the telescopes is chosen so that the pole is reflected into one and an equatorial region (when at an altitude equal to that of the pole) into the other. The polar region will simply rotate round its centre, and the plate must rotate accordingly. The equatorial region will slide perpendicularly to the joining arc, and thus in a vertical direction. Nothing would be gained by moving the mirror. It is quite possible that the polar plate need not be moved at all; the stars will trail, but the motion will be so small that the loss of faint stars will not matter. But if we should come to using as fundamentals *very* faint stars (not because of their faintness, but because of their very small proper motions), then the plate must be made to rotate so as to give them sufficient exposure.

52. *Secondary division.*—The arcs from the pole to the equator can then be bisected just as in R.A., by means of two inclined telescopes and a mirror. But the mirror in this case cannot be a cœlostat mirror, so that it will be necessary to move the plates, unless we take bright stars only and let them trail. We can by the secondary division get the declination of any point in the parallels $\pm 45^\circ$.

53. *Tertiary division.*—The convenient differential method applicable in R.A. has no analogy in filling in the declinations on the same meridian; but when we know the declination of any region, we can find other regions of the same declination by the differential method. We have, however, to show how to find the declinations of points other than those on the equator on the parallel of 45°. It is to be remembered that their R.A.'s may be considered known, and hence we need not find their declination directly, but can deduce it from any arc that involves the declination with fair weight.

54. Consider first a point Q between 0° and -45°. Describe a circle about it of radius 90°, cutting the equator in two points V and W and the parallel of $+45^\circ$ in two points X and Y. We determine the R.A. of Q by measurements of QV or QW. The arcs QX and QY will clearly involve both R.A. and declination. When Q is in declination -45° the points X, Y coincide, and the distance QX measures declination directly. As Q approaches the equator, the angle XQY opens out. If P be the pole, then

$$\cos \mathrm{QX} = \cos \mathrm{QP} \cos \mathrm{PX} + \sin \mathrm{QP} \sin \mathrm{PX} \cos \mathrm{QPX}.$$

But QX is 90° by hypothesis, and PX is 45°. Hence

$$\cot \mathrm{QP} = -\cos \mathrm{QPX},$$

that is to say, Q is in negative declination δ where

$$\tan \delta = \cos QPX.$$

We can thus determine δ directly from the known difference of R.A. of Q and X, or Q and Y.

By choosing the latitude of the observatory we can obviously measure in this way the declinations of points between the parallels $\pm 45^\circ$.

55. For points nearer the pole: to every such point there corresponds another in the same R.A. and 90° different in declination between latitudes $\pm 45^\circ$.

56. Thus determinations of declination of all points are at any rate *possible* by the methods sketched. Whether the actual methods here given are the best and quickest I do not feel sure as yet. A number of possibilities are to be considered. Thus we can easily obtain arcs of 60° as follows: let VW be two points on the equator 90° apart, and X a point on the 45° parallel midway between them in R.A. Then

$$\cos XW = \cos XP \cos WP + \sin XP \sin WP \cos XPW.$$

But $WP = 90^\circ$, $XP = 45^\circ$, and $XPW = 45^\circ$. Thus

$$\cos XW = \tfrac{1}{2},$$

and XW is 60°. We could thus equate it with other arcs, and virtually describe a circle of 60° radius round X, which would give us declinations between 45° and 60°.

57. In practice, no doubt, it will suffice to determine a limited number of fundamental declinations which will check those made in the ordinary way with a circle of some kind. It is only necessary at present to indicate how these may be determined (1) by the measurement of arcs by a fundamental process, and (2) by the comparison of equal arcs, equally affected by refraction; and enough has been said to show the possibilities. Further elaboration may well be deferred until the principles of the method have been given a practical trial.

58. No reference has been made in this paper to the entirely separate problems of determining the position among the stars of the Earth's pole and of the first point of Aries. These were until twenty years ago solved in a crude manner by the transit-circle along with other problems; but after the discovery of the variation of latitude, a number of special instruments were devoted to the former, and the solution of it is practically left to them. The determination of the point where the Sun crosses the equator has not yet been taken up by a special instrument, but it ought to be. The ordinary solution with the transit-circle brings in uncertainties of refraction; probably the Greenwich Altazimuth would really solve the problem better than the transit-circle, since observations

of azimuth near the prime vertical could give the Sun's declination free from these uncertainties. But consideration of these problems cannot be undertaken here. They do not affect the triangulation of the heavens in any way; for if the Earth had not happened to rotate, we could still have surveyed the sky as above, and these two special problems would have had no existence.

On a New Family of Periodic Orbits in the Problem of Three Bodies. By Ernest W. Brown, Sc.D., F.R.S. (Plates 6, 7.)

1. There are four known asteroids which appear to oscillate about one or other of the vertices of the two equilateral triangles which have the line joining Jupiter and the Sun as base. These vertices are the well-known positions of relative equilibrium. The heliocentric vector of one of these asteroids can apparently move as far as 17° away from its equilibrium position.* The oscillations cannot therefore be considered very small. One naturally asks whether oscillations of this kind in arcs of still greater extent are possible; and if so, in what manner the orbits may be most conveniently obtained.

In order to simplify the problem as much as possible, I consider an asteroid of infinitesimal mass, attracted only by the Sun and Jupiter, the three bodies being in one plane. Jupiter will then describe an ellipse about the Sun. For further simplicity, the eccentricity of this ellipse will be neglected, so that Jupiter's motion is circular, and its velocity and distance from the Sun are constant. The problem is therefore the same as that considered by Sir George Darwin in his investigations on periodic orbits.

Take the mass of the Sun as the unit of mass, and the distance of Jupiter from the Sun as the unit of distance. Then, if n be the mean motion of Jupiter and m its mass, $n^2 = 1 + m$.

Let the motion of the asteroid be referred to moving axes with the Sun as origin, the positive direction of the x-axis passing through Jupiter. Let x, y be its coordinates, $r^2 = x^2 + y^2$, and ρ its distance from Jupiter. Then, as Darwin has shown, the equations of motion are †

$$\left.\begin{aligned} \ddot{x} - 2n\dot{y} &= \frac{\partial \Omega}{\partial x}, \quad \ddot{y} + 2n\dot{x} = \frac{\partial \Omega}{\partial y}, \\ \text{with the integral} \qquad \dot{x}^2 + \dot{y}^2 &= 2\Omega - C, \end{aligned}\right\} \tag{1}$$

where

$$2\Omega = \frac{2}{r} + r^2 + m\left(\frac{2}{\rho} + \rho^2\right). \tag{2}$$

* L. J. Linders, *Arkiv för Mat., Ast. och Fys., So. Vet. Ak. i Stockholm*, Bd. 4, No. 20.

† *Acta Mathematica*, vol. xxi. p. 103. The mass of J is there taken as the unit of mass. The slight differences in the equations are easily noted.

We shall need the derivatives

$$\left.\begin{aligned}\frac{\partial^2\Omega}{\partial x^2} &= 1 + m - \frac{1}{r^3} - \frac{m}{\rho^3} + 3\frac{x^2}{r^5} + 3m\frac{(x-1)^2}{\rho^5},\\ \frac{\partial^2\Omega}{\partial y^2} &= 1 + m - \frac{1}{r^3} - \frac{m}{\rho^3} + 3\frac{y^2}{r^5} + 3m\frac{y^2}{\rho^5},\\ \frac{\partial^2\Omega}{\partial x\partial y} &= 3\frac{xy}{r^5} + \frac{3m(x-1)y}{\rho^5}.\end{aligned}\right\} \tag{3}$$

Referred to polar coordinates, with S as pole and SJ as initial line, the equations are

$$\left.\begin{aligned}\ddot{r} - r\dot{\theta}^2 - 2nr\theta &= \frac{\partial\Omega}{\partial r},\\ \frac{d}{dt}(r^2\dot{\theta} + r^2 n) &= \frac{\partial\Omega}{\partial\theta},\\ \dot{r}^2 + r^2\dot{\theta}^2 &= 2\Omega - \mathrm{C}.\end{aligned}\right\} \tag{4}$$

Let ρ, ϕ be polar coordinates with J as pole and the same direction as initial line. The equations referred to these coordinates will be (4), with the letters ρ, ϕ replacing r, θ respectively.

The following relations are needed :—

$$\left.\begin{aligned} r\sin\theta = \rho\sin\phi, \quad r(r - \cos\theta) &= \rho(\rho + \cos\phi),\\ r^2 - 1 &= \rho^2 + 2\rho\cos\phi.\end{aligned}\right\} \tag{5}$$

The derivatives of Ω with respect to r, θ can be expressed as follows :—

$$\left.\begin{aligned}\frac{\partial\Omega}{\partial r} &= r - \frac{1}{r^2} + \frac{m}{r}\left(\rho - \frac{1}{\rho^2}\right)(\rho + \cos\phi),\\ \frac{\partial^2\Omega}{\partial r^2} &= 1 + \frac{2}{r^3} + m\left(1 + \frac{2}{\rho^3}\right) - \frac{3m}{r^2}\frac{\sin^2\phi}{\rho^3},\\ \frac{\partial\Omega}{\partial\theta} &= m\left(1 - \frac{1}{\rho^3}\right)r\sin\theta = m\left(\rho - \frac{1}{\rho^2}\right)\sin\phi,\\ \frac{\partial^2\Omega}{\partial\theta^2} &= m\left(1 - \frac{1}{\rho^3}\right)r\cos\theta + \frac{3m\sin^2\phi}{\rho^3},\\ \frac{\partial^2\Omega}{\partial\theta\partial r} &= m\left(\rho - \frac{1}{\rho^2}\right)\sin\phi + \frac{3m(\rho + \cos\phi)\sin\phi}{r\rho^3}.\end{aligned}\right\} \tag{6}$$

Finally, if V be the velocity relative to the moving axes, R the radius of curvature of the orbit, P the force resolved along the inward normal to the orbit, we have (Darwin, *l.c.*)—

$$\mathrm{V}^2 = 2\Omega - \mathrm{C}, \quad \frac{\mathrm{V}^2}{\mathrm{R}} = \mathrm{P} - 2n\mathrm{V}. \tag{7}$$

The inward normal is defined as that to the left of the orbit, looking in the direction of motion; R is then positive when the orbit is on the left-hand side of its tangent.

2. Let x_0, y_0 be the coordinates of one of the triangular positions of equilibrium. Then $r_0 = \rho_0 = 1$, $x_0 = \frac{1}{2}$, $y_0 = \pm\frac{1}{2}\sqrt{3}$, $C = 3 + 3m$. If we put $x = x_0 + \xi$, $y = y_0 + \eta$, the small oscillations are given by

$$\left.\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \xi\frac{\partial^2\Omega}{\partial x_0^2} + \eta\frac{\partial^2\Omega}{\partial x_0\partial y_0}, \\ \ddot{\eta} + 2n\dot{\xi} &= \xi\frac{\partial^2\Omega}{\partial x_0\partial y_0} + \eta\frac{\partial^2\Omega}{\partial y_0^2}. \end{aligned}\right\} \qquad (8)$$

The solution $\xi = Ae^{\lambda t}$, $\eta = Be^{\lambda t}$ gives for the determination of λ (Darwin, *l.c.*, p. 194),

$$\lambda^4 + n^2\lambda^2 + \frac{27}{4}m = 0.$$

When m is less than ·04 the roots are all imaginary. I am going to suppose that m is very small, having in mind the actual mass of Jupiter for which $m = 1/1048$. One pair of the roots of λ is then nearly equal to $\pm in$, giving a motion in a small ellipse with period nearly equal to that of Jupiter. The other pair is small and nearly equal to $\pm i\frac{3}{2}\sqrt{3m}$, so that the period is long. When $m = 1/1048$, the period is about 13 times that of Jupiter, or about 140 years. It is the latter solution which is under consideration. The former corresponds to a small eccentricity of the orbit, in the sense of the planetary theory.

In order to obtain the positions of the small ellipse which corresponds to each period, turn the axes through a constant angle α. The equations of motion are unaltered, but we now have in the terms not factored by m,

$$x_0 = \cos(\tfrac{1}{3}\pi - \alpha), \quad y_0 = \sin(\tfrac{1}{3}\pi - \alpha);$$

and in those factored by m,

$$x_0 - 1 = \cos(\tfrac{1}{3}\pi - \alpha) - \cos\alpha, \quad y_0 = \sin(\tfrac{1}{3}\pi - \alpha) + \sin\alpha.$$

The angle α is to be so determined that $\partial^2\Omega/\partial x_0\partial y_0 = 0$. This condition gives

$$\mathrm{Tan}(\tfrac{2}{3}\pi - 2\alpha) = -\frac{m\sqrt{3}}{2-m},$$

so that $\alpha \doteq \frac{1}{3}\pi$ * and the new x-axis is nearly along the line joining the Sun to the equilibrium point $\theta = \frac{1}{3}\pi$, $r = 1$.

* For the sake of brevity I use the symbol $\doteq$ throughout to denote "is approximately equal to."

Further,

$$\frac{\partial^2 \Omega}{\partial {x_0}^2} = 3\cos^2(\tfrac{1}{3}\pi - \alpha) + 3m\sin^2(\tfrac{1}{6}\pi - \alpha),$$
$$\frac{\partial^2 \Omega}{\partial {y_0}^2} = 3\sin^2(\tfrac{1}{3}\pi - \alpha) + 3m\cos^2(\tfrac{1}{6}\pi - \alpha),$$

and the equations of variations become

$$\ddot{\xi} - 2n\dot{\eta} = \xi\left(3n^2 - \frac{\partial^2 \Omega}{\partial {y_0}^2}\right),$$
$$\ddot{\eta} + 2n\dot{\xi} = \eta\frac{\partial^2 \Omega}{\partial {y_0}^2},$$

in which $\partial^2\Omega/\partial {y_0}^2$ is a small positive quantity of order m.

We can now assume

$$\xi = A\cos(\lambda t + \alpha), \quad \eta = B\sin(\lambda t + \alpha),$$

then
$$\lambda^4 - n^2\lambda^2 + \frac{27}{4}m = 0,$$

and
$$\frac{A}{B} = -\frac{2n\lambda}{3n^2 - \frac{\partial^2 \Omega}{\partial {y_0}^2} + \lambda^2} = -\frac{\lambda^2 + \frac{\partial^2 \Omega}{\partial {y_0}^2}}{2n\lambda}.$$

For the root $\lambda \doteq \frac{3}{2}\sqrt{3m}$, we have $A/B \doteq -\sqrt{3m}$. Hence one axis of the ellipse is short compared with the other, the shorter axis being directed nearly towards the Sun. The oscillation is therefore very nearly along the circle $r = 1$, the velocity of motion and the deviation from the circle being of the order $\sqrt{m}$ in general compared with the arc described along the circle.

The other root is nearly equal to n and $A/B \doteq -\frac{1}{2}$. The directions of the axes of this ellipse are the same as before. This solution will not be further considered.

3. Any attempt to proceed to higher approximations with rectangular coordinates would be very tedious. We are going to see if it is possible to extend the range of the oscillations in another way. Let us assume that the arc described on or near the circle $r = 1$ is finite, but that $r - 1$, $\dot{r}$, $\dot{\theta}$, etc., are small. In the small oscillations just obtained we find, in comparison with the arc described on the circle $r = 1$, that $r - 1$, $\dot{\theta}$ are of order $\sqrt{m}$, that $\dot{r}$, $\ddot{\theta}$ are of order m, and $\ddot{r}$ of order $m^{\frac{3}{2}}$. Now

$$\dot{r}^2 + r^2\dot{\theta}^2 = 3 - C + 3(r-1)^2 + (r-1)^3 + \cdots + m\left(\frac{2}{\rho} + \rho^2\right).$$

Hence $C - 3$ must be of order m.

The three equations (4) are equivalent to two; let us use the

first and third, and on the assumption just made, neglect all but the two lowest powers of m present in each. We then neglect $\dot{r}^2$ (order m^2) in the third (which is divisible by m), and $\ddot{r}$ (order $m^{\frac{3}{2}}$) in the first (which is divisible by $m^{\frac{1}{2}}$), and obtain

$$r^2\dot{\theta}^2 = 2\,\Omega - C, \quad -r^2\dot{\theta}^2 - 2nr^2\dot{\theta} = r\frac{\partial \Omega}{\partial r}. \tag{9}$$

If we eliminate $\dot{\theta}$ between these equations, we obtain a relation between r, θ which satisfies the differential equations as far as the order laid down. This relation is therefore the equation to the orbit. If r be eliminated we obtain a relation between $\dot{\theta}$, θ which, on integration, furnishes the position at any time.

In order that the process may be justified, it is necessary that the values of $\dot{r}$, $\ddot{r}$ obtained by differentiation from (9) shall be respectively divisible by m, $m^{\frac{3}{2}}$. If this is found to be the case, we can substitute the values of $\dot{r}$, $\ddot{r}$ so found in the first and third of (4) and so obtain a second approximation to the orbit. The process can formally be continued indefinitely.

We are to obtain $\dot{r}$, $\ddot{r}$ from (9); with these equations we may use the second of (4), namely,

$$\frac{d}{dt}(r^2\dot{\theta} + r^2 n) = \frac{\partial \Omega}{\partial \theta}, \tag{10}$$

which is exact, but which is nevertheless deducible from (9).

We have from (9)

$$-2n(r^2\dot{\theta} + r^2 n) = r\frac{\partial \Omega}{\partial r} + 2\,\Omega - C - 2n^2r^2 = \Omega_1 - C, \tag{11}$$

suppose. This, combined with (10), gives

$$-2n\frac{\partial \Omega}{\partial \theta} = \frac{\partial \Omega_1}{\partial r}\dot{r} + \frac{\partial \Omega_1}{\partial \theta}\dot{\theta}, \tag{12}$$

so that

$$\dot{r} = \frac{2n\dfrac{\partial \Omega}{\partial \theta} + \dot{\theta}\dfrac{\partial \Omega_1}{\partial \theta}}{-\dfrac{\partial \Omega_1}{\partial r}}. \tag{13}$$

Again, differentiating (12), I find

$$-2n\frac{\partial^2 \Omega}{\partial r\,\partial \theta}\dot{r} - 2n\frac{\partial^2 \Omega}{\partial \theta^2}\dot{\theta} = \left(\dot{r}^2\frac{\partial^2}{\partial r^2} + 2\dot{r}\dot{\theta}\frac{\partial^2}{\partial r\,\partial \theta} + \dot{\theta}^2\frac{\partial^2}{\partial \theta^2} + \ddot{r}\frac{\partial}{\partial r} + \ddot{\theta}\frac{\partial}{\partial \theta}\right)\Omega_1.$$

Eliminating $\ddot{\theta}$ from this by means of (10), we obtain

$$r = \frac{\left(\dot{r}\dfrac{\partial}{\partial r} + \dot{\theta}\dfrac{\partial}{\partial \theta}\right)^2 \Omega_1 + 2n\left(\dot{r}\dfrac{\partial}{\partial r} + \dot{\theta}\dfrac{\partial}{\partial \theta}\right)\dfrac{\partial \Omega}{\partial \theta} - \left(2r\dot{r}\dot{\theta} + 2r\dot{r}n - \dfrac{\partial \Omega}{\partial \theta}\right)\dfrac{1}{r^2}\dfrac{\partial \Omega_1}{\partial \theta}}{-\dfrac{\partial \Omega_1}{\partial r}}. \tag{14}$$

This, on the substitution of the value (13) of $\dot{r}$, gives $\ddot{r}$ in the required form.

Since $$-\frac{\partial \Omega_1}{\partial r} = 4n^2 r - r\frac{\partial^2 \Omega}{\partial r^2} - 3\frac{\partial \Omega}{\partial r}$$

$$= \frac{1}{r^3} + \frac{mr}{\rho^3} + 3m\cos\theta\left(1 - \frac{1}{\rho^3}\right) + \frac{3m\sin^2\phi}{r\rho^3},$$

the common denominator of (13), (14) is nearly unity as long as ρ is not small. And since all derivatives with respect to θ of Ω, Ω_1 have the factor m, it follows that $\dot{r}$ has the factor m and $\ddot{r}$ the factor $m^{\frac{3}{2}}$.

The original assumption is therefore justified. The limitation placed on the process is that ρ shall not be small. It is obvious that the process is then convergent for sufficiently small values of m. It fails when ρ becomes of the order of magnitude $m^{\frac{1}{3}}$. For then $\partial \Omega_1/\partial r$ will vanish for values of ϕ not very near $\pi/2$.

Since $r^2 - 1 = \rho^2 + 2\rho\cos\phi$ and $r = 1$, it follows that if ρ is of order $m^{\frac{1}{3}}$, $r - 1$ is of the same order. The original hypothesis—that $r - 1$ is of order $m^{\frac{1}{3}}$—no longer holds. In fact expansions in powers of $m^{\frac{1}{3}}$ are probably no longer convergent. It will be necessary to expand in powers of ρ, and the series is therefore one in powers of $m^{\frac{1}{3}}$. This is in accordance with the usual satellite problem where we expand the disturbing function in powers of the ratio of the distances of the satellite and of the Sun from the planet.

4. The form which constitutes the second approximation can be obtained without the neglect of $\dot{r}^2$ in the Jacobian integral.

We have

$$\dot{\theta}^2 + 2n\dot{\theta} = -\frac{1}{r}\frac{\partial \Omega}{\partial r} + \frac{\ddot{r}}{r},$$

and therefore

$$2\ddot{\theta}(\dot{\theta} + n) = -\dot{r}\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Omega}{\partial r}\right) - \dot{\theta}\frac{1}{r}\frac{\partial^2 \Omega}{\partial r \partial \theta} + \frac{d}{dt}\left(\frac{\ddot{r}}{r}\right).$$

Also $$r^2\ddot{\theta} = \frac{\partial \Omega}{\partial \theta} - 2r\dot{r}(\dot{\theta} + n).$$

Whence, eliminating $\ddot{\theta}$ between these two equations, we obtain

$$\dot{r}\left[r\frac{\partial^2 \Omega}{\partial r^2} - \frac{\partial \Omega}{\partial r} - 4r(\dot{\theta} + n)^2\right] = -2(\dot{\theta} + n)\frac{\partial \Omega}{\partial \theta} - r\frac{\partial^2 \Omega}{\partial r \partial \theta}\cdot\dot{\theta} + r^2\frac{d}{dt}\left(\frac{\ddot{r}}{r}\right).$$

But, from the first of the above equations,

$$r(\theta + n)^2 = -\frac{\partial \Omega}{\partial r} + \ddot{r} + rn^2;$$

hence

$$\dot{r}\left[4rn^2 - r\frac{\partial^2 \Omega}{\partial r^2} - 3\frac{\partial \Omega}{\partial r}\right] = \dot{\theta}\left(r\frac{\partial^2 \Omega}{\partial r \partial \theta} + 2\frac{\partial \Omega}{\partial \theta}\right) + 2n\frac{\partial \Omega}{\partial \theta} - r^2\frac{d}{dt}\left(\frac{\ddot{r}}{r}\right) - 4\dot{r}\ddot{r}.$$

If we neglect $\ddot{r}$, $\dddot{r}$, these two equations will give $\dot{r}$, $\dot{\theta}$ in terms of r, θ, and thence, in combination with $\dot{r}^2+r^2\dot{\theta}^2=2\Omega-C$, the equation to the orbit. The second equation, when $\ddot{r}$, $\dddot{r}$ are neglected, is the same as (13)—that previously found for $\dot{r}$.

We have therefore no need to suppose that $\dot{\theta}$ is small. Whatever plan be adopted for approximation, the value (14) previously obtained for $\ddot{r}$ shows that $\ddot{r}$ is divisible by m, so long as ρ is not small.

5. In order to get the forms of the orbits, consider the first approximation given by eliminating $\dot{\theta}$ between

$$r^2\dot{\theta}^2=2\Omega-C, \qquad -r^2\dot{\theta}^2-2nr^2\dot{\theta}=r\frac{\partial\Omega}{\partial r}.$$

With the original assumption that $r\dot{\theta}$ was small and with $\dot{r}$ also small, the orbits are all characterised by small velocity. Hence any orbit cannot be far from its zero velocity curve, and since $2\Omega-C$ is positive, it must cross only those members of the family $2\Omega-C'=0$ for which $C'>C$.

In the first approximation the orbit touches its zero velocity curve where $\partial\Omega/\partial r=0$, that is, where r is tangent to that curve. In the higher approximations there is no contact, but only a close approach.

An orbit touches one of the curves of equal velocity in four or six places only. For, at such a point of contact, we have

$$\frac{\dot{\theta}}{\dot{r}}=-\frac{\frac{\partial\Omega}{\partial r}}{\frac{\partial\Omega}{\partial\theta}}\doteq\frac{r\dot{\theta}^2+2nr\dot{\theta}}{\frac{\partial\Omega}{\partial\theta}}.$$

This is satisfied by $\dot{\theta}=0$ or by

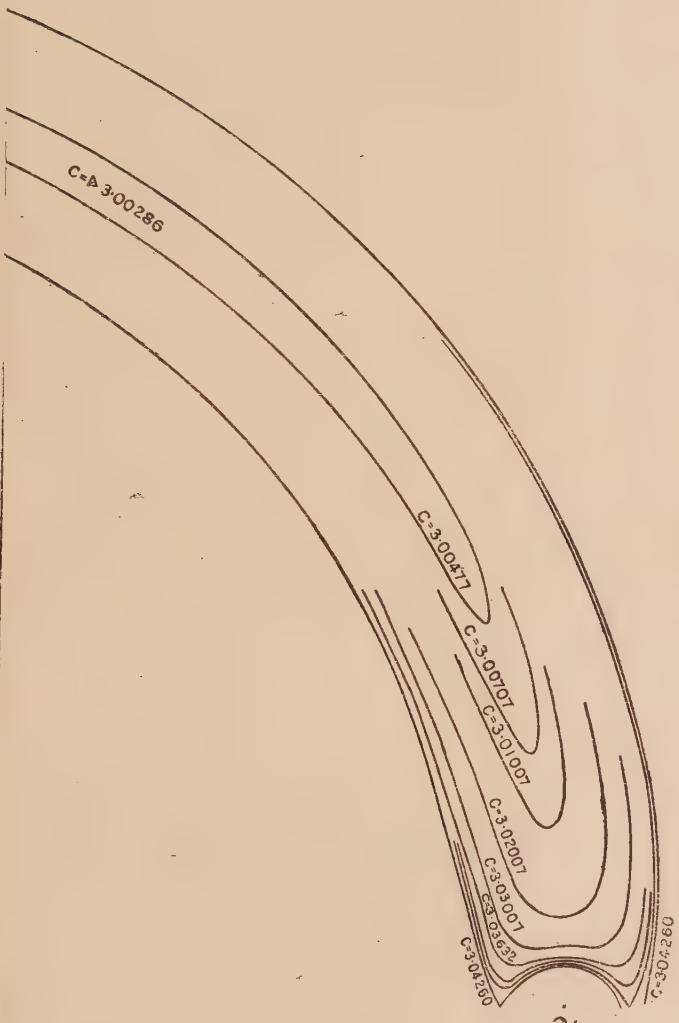
$$\frac{\partial\Omega}{\partial\theta}=(2nr+r\dot{\theta})\dot{r}\doteq 2nr\dot{r}\doteq 2nr\frac{\dot{\theta}\frac{\partial\Omega_1}{\partial\theta}+2n\frac{\partial\Omega}{\partial\theta}}{-\frac{\partial\Omega_1}{\partial r}}\doteq -4n^2\frac{\partial\Omega}{\partial\theta},$$

so that $\partial\Omega/\partial\theta\doteq 0$. This can only be satisfied by $\rho\doteq 1$ or $\theta\doteq 0$, π.

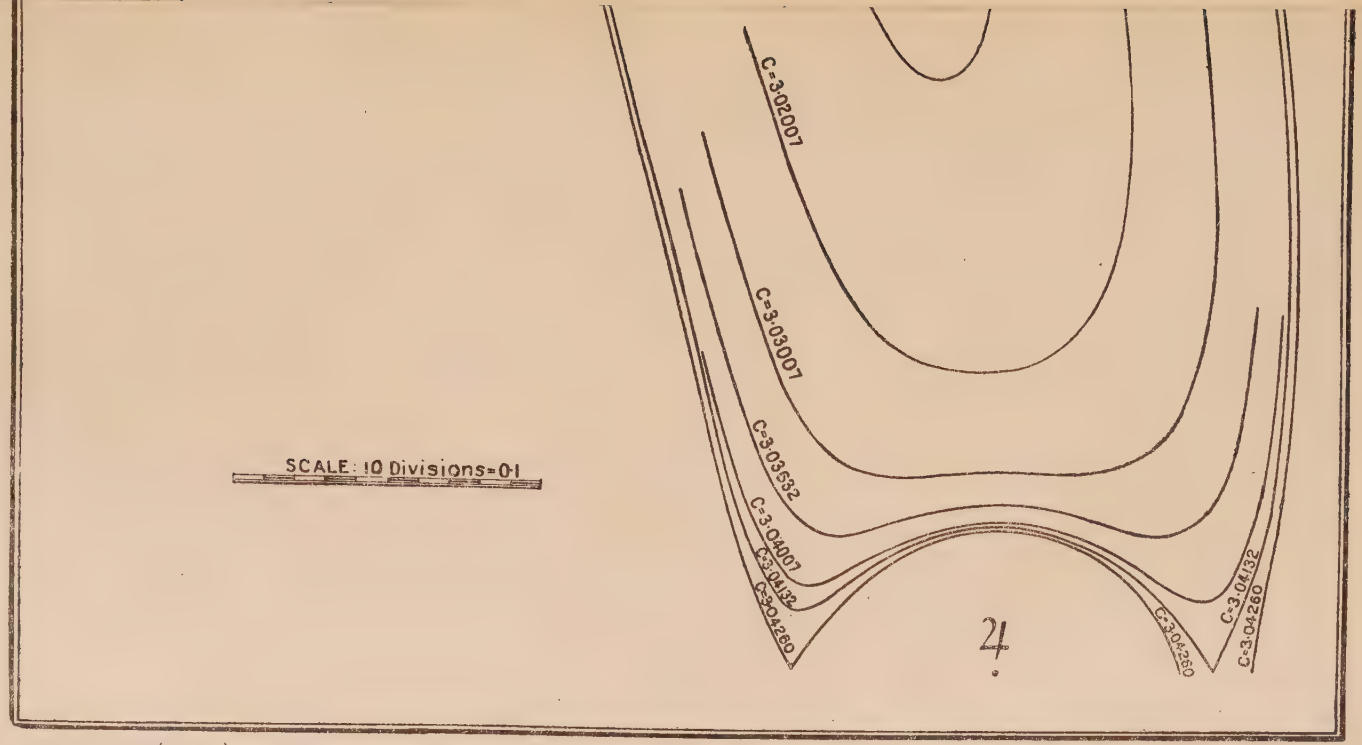
We have excluded small values of ρ and therefore of θ. If, then, θ cannot reach the value π, we have two places near where $\dot{\theta}=0$ and two where $\rho\doteq 1$; the former are places of closest approach to the zero velocity curve, and the latter places of greatest distance from this curve. If θ can reach the value π, we have two more places of closest approach, which occur when $\theta=\pi$ exactly.

It is then sufficient, in order to obtain a good idea of the orbits, to trace the zero velocity curves and to find points on the orbit where $\rho\doteq 1$, $\dot{\theta}\doteq 0$, $\theta=\pi$.

As soon as we attempt to compute with the method just



Computed and drawn by Henry B. Hedrick.



$\frac{2}{r}+r^2+m\left(\frac{2}{\rho}+\rho^2\right)=C.$ $\odot$ to $\mathcal{U}=1.$ $m=\frac{1}{1048}.$

Computed and drawn by Henry B. Hedrick.

given, it is seen that there will be no failure as long as ρ is not small. We can, therefore, obtain all the orbits in which the asteroid does not approach Jupiter very closely. The limiting value of ρ for purposes of computation is about ·25, when $m = 1/1048$.

6. The zero velocity curves have been drawn with great care and accuracy by Mr. H. B. Hedrick and are shown on Plate 6. In Plate 7 the portion of them near Jupiter is given by him on an enlarged scale, so that the changes near the equilibrium points may be clearly seen. They are, of course, similar to those obtained by Darwin (*l.c.*) for $m = 1/10$, but here the horse-shoe curves are very narrow and the ratio of the bulbs of the hour-glass curves is quite small.

Mr. Hedrick gives the following values for the coordinates of the equilibrium points and the corresponding values of C :—

$$\begin{aligned}
&\rho = r = 1, && C = 3{\cdot}00286 = 3 + 3m,\\
&\rho = r + 1 = 1{\cdot}99944, && C = 3{\cdot}00477 \doteq 3 + 5m,\\
&\rho = r - 1 = {\cdot}06977, && C = 3{\cdot}04132 \doteq 3 + 3^{\frac{1}{3}}m^{\frac{2}{3}} - \tfrac{2}{3}m,\\
&\rho = 1 - r = {\cdot}06667, && C = 3{\cdot}04260 \doteq 3 + 3^{\frac{1}{3}}m^{\frac{2}{3}} + \tfrac{2}{3}m.
\end{aligned}$$

They have been verified by Darwin's approximate formulæ (*l.c.*, § 3) which are sufficiently accurate for the purpose when $m = 1/1048$, and which are given above as far as the order m.

The value $C = 3 + 5m$ corresponds to the case of the limiting horse-shoe curves. It appears that the assumption, previously made in order that the method of approximation to the orbit may be possible (that $C - 3$ be divisible by m), should hold for horse-shoe orbits, and indeed for orbits corresponding to considerably larger values of C. If we put $C = 3 + km$, it should hold until k becomes comparable with the value $3^{\frac{1}{3}}/m^{\frac{1}{3}} \doteq 40$ when $m = 1/1048$. There is therefore a considerable range of the horse-shoe orbits which can be computed by the method.*

The stationary values of ρ in the zero velocity curves are at the places where $r = 1$ or $\phi = 0, \pi$. The smallest value of ρ for $C = 3 + 5m$ is found by Mr. Hedrick to be ·414. Incidentally, he found the radius of curvature at this place to be ·0022. The turn in this and all the zero velocity curves for values of C less than $3 + 5m$ is therefore very sudden.

7. *The limiting horse-shoe orbit.* — Here $C \doteq 3 + 5m$. Mr. Hedrick finds that at the place of minimum distance of the zero velocity curve from J,

$$r = 1{\cdot}000, \quad \rho = {\cdot}414, \quad \theta = \pm 23^{\circ}{\cdot}9.$$

This, to the degree of approximation given, is also the minimum value of θ.

* This result was announced in my address before Sect. A of the Amer. Assoc. Adv. Sc., 1910 Dec. 28, see *Science*, 1911 Jan. 20.

In order to obtain the minimum value of θ in the *orbit*, we use these values as a first approximation, and compute the values of $\dot{r}$, $\ddot{r}$ from the formulæ given in No. 3. I find

$$\dot{r} = -\cdot0099, \quad \ddot{r} = +\cdot000031.$$

With these, a second approximation is obtained, giving

$$\rho = \cdot405, \quad \theta = \pm 23^{\circ}\cdot4,$$

the value of r not being sensibly changed. This is then the position of nearest approach of the asteroid to Jupiter.

The value $\rho = 1$ gives very nearly the position of greatest separation of the orbit from its zero velocity curve. With the first approximation I find, for the outer portion of the orbit,

$$\rho = 1, \quad r\dot{\theta} = -\cdot0740, \quad r = 1\cdot0500.$$

With these values substituted in (13) and (14),

$$\dot{r} = -\cdot00011, \quad \ddot{r} = -\cdot00044,$$

and thence the second approximation,

$$\rho = 1, \quad r\dot{\theta} = -\cdot0749, \quad r = 1\cdot0504.$$

A very small change in r will reduce $\dot{r}$ to zero. This is then very nearly also the position of maximum distance of the asteroid from the Sun.

The value $\rho = 1$ also gives very nearly a minimum value of r for the inner portion of the orbit. I find that $r = \cdot95$, while $\dot{r}$, $r\theta$ have approximately the same numerical values as at the maximum distance, but with the opposite signs.

Finally, the orbit passes through the equilibrium point in opposition to Jupiter. It has a node at this point, and the two tangents are inclined to the x-axis at angles which differ from right angles by about 1°.

8. Let us examine more closely the changes taking place near the opposition equilibrium point, as the series passes from oscillations in which $|\theta| < \pi$ to the horse-shoe orbits. For values of C a little less than the critical value there are two orbits, the ends of which closely approach the point; for values a little greater, there are two orthogonal intersections of the horse-shoe orbit with the x-axis which are also close to the point. I shall show that *the portions of these orbits in the neighbourhood of the equilibrium point correspond to the unstable solutions which are obtained when we form the equations for small oscillations about the position of equilibrium.*

Let r_0 be the distance of the point from S. Then $\partial^2 \Omega / \partial x_0 \partial y_0 = 0$,

$$\frac{\partial^2 \Omega}{\partial x_0^2} = 1 + m + \frac{2}{r_0^3} + \frac{2m}{(1+r_0)^3},$$
$$\frac{\partial^2 \Omega}{\partial y_0^2} = 1 + m - \frac{1}{r_0^3} - \frac{m}{(1+r_0)^3} = -\beta^2,$$

the value for r_0 given in No. 6 showing that $\partial^2 \Omega / \partial y_0^2$ is negative. The equations (8) for small oscillations become

$$\left.\begin{aligned} \ddot{\xi} - 2n\dot{\eta} &= \xi(3n^2 + 2\beta^2), \\ \ddot{\eta} + 2n\dot{\xi} &= -\beta^2\eta. \end{aligned}\right\} \quad (15)$$

If we put $\xi = Ae^{\lambda t}$, the equation for λ is

$$\lambda^4 + \lambda^2(n^2 - \beta^2) - \beta^2(3n^2 + 2\beta^2) = 0.$$

There are therefore two pairs of solutions. Since β^2 is small (of order m), one of these is imaginary and gives motion in an ellipse with a period which is approximately $2\pi/n$; the other is real and corresponds to the approximate value $\lambda^2 = 3\beta^2$. For this latter pair, following Hough's method of treatment,* I put

$$\xi = q(a_1 e^{\lambda t} + a_2 e^{-\lambda t}),$$
$$\eta = a_1 e^{\lambda t} - a_2 e^{-\lambda t},$$

where

$$\frac{1}{q} = \frac{\lambda^2 - 3n^2 - 2\beta^2}{2n\lambda} = \pm \frac{\sqrt{3}}{2\beta}.$$

The equation to the curve is

$$\frac{\xi^2}{q^2} - \eta^2 = 4a_1 a_2.$$

It is therefore an hyperbola with the real axis along or perpendicular to the x-axis according as $a_1 a_2$ is positive or negative. Also ξ/a_1, $\dot{\eta}/a_1$ are of order $m^{\frac{1}{3}}$; and since λ is of order $m^{\frac{1}{3}}$, we have $\dot{\xi}/a_1$, $\ddot{\eta}/a_1$ of order m, and $\ddot{\xi}/a_1$ of order $m^{\frac{2}{3}}$.

But the equations (15) are also those of the orbits near the position of equilibrium. The method of solution adopted was simply one of continued approximation along powers of $m^{\frac{1}{3}}$, beginning with that solution obtained by neglecting r, which, in this case, is the same as ξ. We must obviously obtain the same result whichever method we adopt. Hence the statement.

We easily find

$$C = 2\Omega_0 + 2a_1 a_2 \mu = 3{\cdot}00477 + 2a_1 a_2 \mu,$$

where

$$\mu = q^2(3n^2 + 2\beta^2 - \lambda^2) - \lambda^2 - \beta^2 = \text{pos. quantity}.$$

* *Acta Math.*, vol. xxiv. p. 260.

Hence C is greater than the critical value when $a_1 a_2$ is positive, and less when it is negative, in agreement with the values for the orbits. It is easily seen that the directions of motion agree also with those in the orbits. There is thus no discontinuity in the family of orbits while C passes through the critical value.

9. The equations of the orbits having been found, we have now to show how the position at any time may be obtained. Consider the first approximation given by

$$r^2\dot{\theta}^2 = \frac{2}{r} + r^2 + m\left(\frac{2}{\rho} + \rho^2\right) - \mathrm{C},$$

$$-\dot{r}^2 - 2nr^2\dot{\theta} = -\frac{1}{r} + r^2 + m\left(1 - \frac{1}{\rho^3}\right)(r - \cos\theta).$$

In order to find a relation between θ, t, it is necessary to eliminate r. For this purpose methods of approximation must be used. We have

$$\rho^2 = 1 - 2r\cos\theta + r^2 = 4r\sin^2\frac{\theta}{2} + (1-r)^2.$$

As long as $1-r$ is small compared with $2\sin\frac{\theta}{2}$, we can expand in powers of $s^2 = (1-r)^2$. We must take $\sqrt{\sin^2\frac{\theta}{2}}$ with the positive sign.

Put $\mathrm{C} = 3 + c$. Then, for those orbits which can be computed by the method, c is of order m. Hence the first equation (which is divisible by m) becomes, if we neglect quantities of order higher than m,

$$\dot{\theta}^2 = 3s^2 + m\left(\frac{1}{\sin\frac{\theta}{2}} + 4\sin^2\frac{\theta}{2}\right) - c.$$

The terms in the second equation are divisible by $\sqrt{m}$. Neglecting all powers of m but the lowest present, we obtain $2\dot{\theta} = -3s$. Hence

$$\dot{\theta}^2 = 3c - 3m\left(\frac{1}{\sin\frac{\theta}{2}} + 4\sin^2\frac{\theta}{2}\right),$$

and

$$t + a = \int \frac{d\theta}{\sqrt{3c - 3m\left(\frac{1}{\sin\frac{\theta}{2}} + 4\sin^2\frac{\theta}{2}\right)}}.$$

This is a periodic function. The maxima and minima of the part under the square root are given by $\theta = 180°$, $60°$, $300°$, provided these values do not make $\dot{\theta}$ imaginary. The value

$\theta = 180°$ gives $\dot{\theta}^2 = 3(c - 5m)$, and the values $\theta = 60°$, $300°$ give $\dot{\theta}^2 = 3(c - m)$.

Hence $c > 3m$; this is known otherwise, for $C = 3 + 3m$ is the least value of C for which these solutions exist.

If $3m < c < 5m$, the limiting values of θ are given by

$$\sin^3 \frac{\theta}{2} - \frac{c}{4m} \sin \frac{\theta}{2} + \frac{1}{4} = 0.$$

There is one value of $\sin \frac{1}{2}\theta$ greater than unity, the other two values lie between 1, $\frac{1}{2}$ and $\frac{1}{2}$, 0. We then have oscillations about the triangular equilibrium points in which $|\theta|$ is always less than π and greater than zero.

If $c > 5m$, there is only one value of $\sin \frac{\theta}{2}$ which is less than unity, and that lies between $\frac{1}{2}$, 0. This case is that of the horse-shoe orbits.

If θ_0, θ_1 be the extreme values of θ when $c < 5m$, the period is given by

$$T = 2 \int_{\theta_0}^{\theta_1} \frac{d\theta}{\sqrt{3c - 3m\left(\frac{1}{\sin \frac{\theta}{2}} + 4 \sin^2 \frac{\theta}{2}\right)}}.$$

If $c > 5m$, the period is

$$T = 4 \int_{\theta_0}^{\pi} \frac{d\theta}{\sqrt{3c - 3m\left(\frac{1}{\sin \frac{\theta}{2}} + 4 \sin^2 \frac{\theta}{2}\right)}}.$$

When $c = 5m$, the integral is discontinuous, the discontinuity being of the form $A \log x$ where $x = 0$.

The numerical computation for the position of an asteroid at any time, where θ differs considerably from 60°, will probably have to be made by mechanical quadratures. The various well-known devices for this purpose need not be insisted on in this place.

The method can obviously be applied to the higher approximations. We obtain for $\dot{\theta}^2$ a series in powers of $\sqrt{m}$, beginning with m, and having coefficients which are functions of θ.

10. The stability of the orbits must be considered. Darwin (*l.c.*) has shown that the small normal deviation p from any orbit is given by

$$\ddot{p} + \Theta p = 0,$$

where

$$\Theta = \frac{1}{r^3} + \frac{m}{\rho^3} - \frac{3}{r^3} \cos^2(\psi - \theta) - \frac{3m}{\rho^3} \cos^2(\psi - \phi) + 3\left(\frac{V}{R} + n\right)^2.$$

Here R is the radius of curvature in the periodic orbit, $\dot{x} = -V\sin\psi$, $\dot{y} = V\cos\psi$, so that Θ is a function of the co-ordinates and the time in the periodic orbit.

We can find an approximate value for Θ at any point in the orbit. We have

$$\frac{V}{R} = \frac{r^2\dot{\theta}^3 + 2\dot{r}^2\dot{\theta} - r\ddot{r}\dot{\theta} + r\dot{r}\ddot{\theta}}{V^2} \doteq \frac{\dot{r}\ddot{\theta} - \ddot{r}\dot{\theta}}{V^2},$$

since $\dot{r}$, $r\dot{\theta}$ are small, and $\dot{r}/V$, $r\dot{\theta}/V$ are equal to or less than unity. Again, from the expressions in No. 3 for $\dot{r}$, $\ddot{r}$, $\ddot{\theta}$, we obtain by the use of the terms of lowest order with respect to m,

$$\dot{r} \doteq 2\frac{\partial\Omega}{\partial\theta}, \quad \ddot{\theta} \doteq -\frac{3}{2}\dot{r}, \quad \frac{\ddot{r}\dot{\theta}}{V^2} \doteq 0,$$

$$\frac{\dot{r}\ddot{\theta}}{V^2} \doteq -\frac{3}{2}\frac{\dot{r}^2}{V^2} = -\frac{3}{2}\sin^2(\psi-\theta).$$

Hence, neglecting terms having the factor m,

$$\Theta \doteq 1 - 3\cos^2(\psi-\theta) + 3\left\{1 - \frac{3}{2}\sin^2(\psi-\theta)\right\}^2$$

$$= \left\{1 - \frac{3}{2}\sin^2(\psi-\theta)\right\}\left\{1 - \frac{9}{2}\sin^2(\psi-\theta)\right\}.$$

In the periodic orbits $|\psi-\theta| \doteq 0$ except near the places where $\dot{\theta}$ becomes zero; hence $\Theta \doteq 1$ in general. Now Θ is always positive except when $\sin^2(\psi-\theta)$ lies between $2/3$, $2/9$, or when $|\psi-\theta|$ lies between $55°$, $28°$. It is therefore only in these intervals that the stability is doubtful. The difficulty arises from the fact that the orbit takes a sharp turn where $\dot{\theta} = 0$. At all other points of the orbit $\Theta \doteq 1$; near the turn it may approach the minimum value $-1/3$. We can, however, examine the question in another way.

If we put $s = \log r$, the equations (4) may be written

$$\left.\begin{aligned} \ddot{s} - 2n\dot{\theta} &= \frac{1}{r}\frac{\partial\Omega}{\partial r} + \frac{2\Omega - C}{r^2} = P, \\ \ddot{\theta} + 2n\dot{s} + 2\dot{s}\dot{\theta} &= \frac{1}{r^2}\frac{\partial\Omega}{\partial\theta} = Q. \end{aligned}\right\} \qquad (16)$$

Form the equations of variations by putting $s+\delta s$ for s and $\theta+\delta\theta$ for θ, so that s, θ will now represent coordinates in the periodic orbit. We obtain

$$\left.\begin{aligned} \frac{d^2}{dt^2}\delta s - 2n\frac{d}{dt}(\delta\theta) &= \frac{\partial P}{\partial s}\delta s + \frac{\partial P}{\partial\theta}\delta\theta, \\ \frac{d^2}{dt^2}\delta\theta + 2n\frac{d}{dt}(\delta s) &= \frac{\partial Q}{\partial s}\delta s + \frac{\partial Q}{\partial\theta}\delta\theta - 2\dot{s}\frac{d}{dt}(\delta\theta) - 2\dot{\theta}\frac{d}{dt}\delta s. \end{aligned}\right\} \qquad (17)$$

Let us attempt to solve these linear equations by approximation along powers of $m^{\frac{1}{3}}$, neglecting at first all powers of this parameter. Then $\partial P/\partial s = 3$; the other derivatives of P, Q vanish, as do $\dot{s}$, $\dot{\theta}$. The equations reduce to

$$\left.\begin{aligned}\left(\frac{d^2}{dt^2} - 3\right)\delta s - 2\frac{d}{dt}\delta\theta = 0,\\ \frac{d^2}{dt^2}\delta\theta + 2\frac{d}{dt}\delta s = 0.\end{aligned}\right\} \qquad (18)$$

The latter gives

$$\frac{d}{dt}(\delta\theta) + 2\delta s = 2a_1$$

and thence from the former,

$$\delta s = 4a_1 + 2a_2 \sin(nt + a_3),$$

where a_1, a_2, a_3 are constants of integration.

Now the complete solution of the equations is the sum of four independent solutions, and two of these can always be exhibited as the solutions obtained by varying the two constants present in the periodic orbit. As the stability question involves the supposition that the constants of the periodic orbit remain unchanged, we must choose the constants a_i in such a manner as to fulfil this condition.

The first approximation to the periodic orbit is

$$\dot{\theta} = -\tfrac{3}{2}(r - 1) = -\tfrac{3}{2}s$$

to the required degree of accuracy. Hence if we vary the constants of the periodic orbit we have the relation

$$\frac{d}{dt}\delta\theta = -\frac{3}{2}\delta s.$$

Comparing with the previous values for $\delta\theta$, δs, we find that $a_2 = 0$ and $\delta s = 4a_1$. If, then, the constants of the periodic orbit be not varied, we can place $a_1 = 0$. The constant arising with the further integration of the last equation can be similarly treated and placed equal to zero. Hence the first approximation to the required solution is

$$\delta\theta = 4a_2 \cos(nt + a_3), \quad \delta s = 2a_2 \sin(nt + a_3).$$

The further approximations proceed in the same manner if we adopt the usual device of changing n into $n + \delta n$, and so determine δn that secular terms are not present. The reduction of this solution to forms convenient for computation appears likely to be very troublesome when the extreme value of $|\theta|$ approaches or

becomes equal to π. It may be better to use the equation (17) for limited portions of the periodic orbit, always choosing the solution with period nearly equal to n.

11. The continuation of the orbits for values of C which give minima values of ρ less than about one-quarter does not at present seem to be possible with the previous methods. There are, however, several considerations which lead to the conjecture that the orbits will still follow the general course of the zero velocity curves.

In the first place, it is still possible to use the method for orbits started with any value of C as long as we confine ourselves to those parts of the orbits for which ρ is greater than $\cdot 25$.

Secondly, all these orbits have places of maximum distance from their respective zero velocity curves somewhere in the neighbourhood of the places where $\rho = 1$, and are approaching them from these places up to positions as near Jupiter as $\rho = \cdot 25$. There is therefore an odd number of minima of approach in the portion of the orbit enclosed by the circle $\rho = \cdot 25$. For those orbits (previously considered) which do not cut this circle there is but one minimum of approach.

Thirdly, if the orbits still exist for the critical values $C = 3 \cdot 04132$ and $C = 3 \cdot 04260$ (the values for which the zero-velocity curves pass through the equilibrium points in the neighbourhood of Jupiter), the corresponding orbits must pass through those equilibrium points. Further, the neighbouring orbits correspond to the unstable solutions round those points exactly in the manner shown in No. 8 for the opposition equilibrium point.

Fourthly, the orbits for values of C well beyond $C = 3 \cdot 04260$ can be computed by the method. They consist (1) of an inner planetary orbit making complete revolutions round the Sun in the positive sense; (2) of an outer planetary orbit making complete revolutions round the Sun in the negative sense relative to the moving axes; (3) of a satellite revolving round Jupiter in the positive sense. The last is computed with the ρ, ϕ equations referred to Jupiter as origin.

Fifthly, when the orbit reaches places where the force is very small, it either approaches very closely to the zero velocity curve or the radius of curvature becomes small. For if, in the equation $V^2/R = P - 2nV$, we have P very small, then either V is very small or $R \doteq -2nV$. The latter, as V has been diminishing towards such places, demands that R be small and negative. The orbit should therefore take a rapid turn at the places where the force is small; Mr. Hedrick's figure shows that these are precisely the places where the velocity curves have sharp turns.

Finally, the analytical continuity of such a series of orbits must be maintained.

12. As an illustration, let us take the value $C = 3 \cdot 0420$ which lies between the two critical values mentioned in the last number, and which should therefore give two orbits, the first of hour-glass form enclosing the Sun and Jupiter, the second of nearly circular

form, with motion in the negative sense relative to the moving axes, and lying entirely outside the circle $r = 1$.

For the hour-glass orbit I computed the following quantities:—

The point in opposition to Jupiter. Here $\theta = 180^\circ$. The first approximation gave

$$\rho = r + 1 = 1{\cdot}8022, \quad r\dot{\theta} = {\cdot}3144.$$

Hence

$$\dot{r} = 0, \quad \ddot{r} = -{\cdot}0006,$$

furnishing the second approximation,

$$\rho = r + 1 = 1{\cdot}8026, \quad r\dot{\theta} = {\cdot}3150.$$

The place $\rho = 1$. Here the first approximation gave

$$r = {\cdot}800, \quad r\dot{\theta} = +{\cdot}318,$$

which would scarcely be changed by the second approximation.

For the superior planetary orbit, the first approximation gave

$$\text{for } \theta = \pi: \quad \rho = r + 1 = 2{\cdot}249, \quad r\dot{\theta} = -{\cdot}353;$$
$$\text{for } \rho = 1: \quad r = 1{\cdot}251, \quad r\dot{\theta} = -{\cdot}354.$$

It is significant that in both orbits the velocity perpendicular to the radius vector has scarcely changed during two-thirds of the angular change of position. We have in these orbits an illustration of the use of the method when $r\dot{\theta}$ is not very small but $\dot{r}$, $\ddot{r}$, etc., are very small.

13. If the considerations of No. 11 justify us in concluding that the orbits exist as there conjectured, we have a complete family. Starting from infinitely small orbits round the triangular points, which expand mainly along the circle $r = 1$ until they join to form horse-shoe orbits, the series progresses by the opposite ends of the horse-shoe joining, so that the orbits separate into an outer planetary orbit and an inner hour-glass orbit, the latter afterwards separating into an inner planetary orbit and a satellite orbit. The final stage as C approaches infinity is an orbit of infinite radius and orbits of zero radius round the Sun and Jupiter.

The classification of the family according to values of C is not, however, one which is natural in the consideration of the arrangement of the solar system. If we regard the orbits from the point of view of the mean distance from the Sun, they show a mode of transition, perhaps the most probable, for bodies moving in orbits superior to the planet to bodies in orbits inferior to the planet. It is seen that there are four possibilities. The superior planet may become an inferior planet, a satellite of Jupiter, or a planet oscillating about either of the two triangular equilibrium points, without any break in the continuity of the orbits. From this point of view, one would ask for the continuation after the orbits round the equilibrium points have become infinitely small.

It would be natural to consider that they pass into the other pair of (stable) solutions round these points. The further development of the latter is unknown. In view of the orbits computed by Sir George Darwin * for values of C less than $3+3m$ when $m=1/10$, it is possible that they elongate towards the Sun, finally becoming orbits of ejection.

There are further questions which require examination. Although I believe that the orbits are, in a physical sense, stable, it will be necessary to consider separately the cases in which the deviation is infinitesimal and those in which it is of the order $\sqrt{m}$ or greater. It appears from No. 10 that the period of the deviation is very nearly that of Jupiter, so that we should naturally regard the deviation as that produced by an eccentricity of the orbit of the asteroid in the sense of the planetary theory. I have reason to suppose that this eccentricity may be compelled to lie between limits which are much closer than those for the ordinary planetary orbit if stability is to be maintained. But these matters, as well as the consideration of the history of the orbits when $m>1/25$, I must lay aside for the present.

New Haven, Conn., U.S.A.:
1911 *February* 7.

Second Note on Multiple Solutions in the Determination of Orbits from Three Observations. By Professor C. V. L. Charlier, Assoc. R.A.S.

In my communication of November last I have shown that the problem of determining the orbit of a planet or comet from three complete observations leads to two real solutions when the planet (or comet), at the time of the observations, is situated in the regions denoted as A and C, whereas the solution is unequivocal in the regions denoted as B and D. The words I have added at the end of my letter to the Secretaries on the method of Olbers have, however, been rather misinterpreted,—very naturally, indeed, when we consider that they stand in no direct connection with the previous investigation.

Supposing it to be known or postulated, as in the method of Olbers, that the unknown orbit of a comet is exactly a parabola, then we have only 5 elements to determine, whereas the three complete observations place 6 independent data at our disposal. Hence the problem is over-determined. We conclude that the problem of determining a *parabolic* orbit from three observations, at least *generally*, ought to be unequivocal. This has indeed been pointed out already by Laplace (1780) and by Olbers (1797). The actual elaboration of a *linear* equation for the distance of the comet has been performed by Luc Picart (*Bull. Astr.*, 1899).

* *M.N.*, *R.A.S.*, vol. lxx. pp. 133, 134.

Though the solution of this problem is *generally* unequivocal, it seems at first sight possible that double (or multiple) solutions might occasionally occur; the necessary condition being that the two algebraical equations (2) and (5*a*) below shall have two (or more) common real roots. A closer investigation, however, shows that this *never can occur*. This remarkable fact has been first discovered by Tscherny (*A.N.*, bd. 176 (1907)). He derives the result from a discussion of an approximate form of the equation of Euler—

$$\{r_1 + r_3 + s\}^{3/2} - \{r_1 + r_3 - s\}^{3/2} = 6k(t_3 - t_1) \quad . \quad . \quad (1)$$

compared with the equation of Lagrange (formula (2) of my preceding note).

Hence the determination of a parabolic orbit from three complete observations can be performed only in one single manner. Nevertheless there are found sometimes, in the literature of the subject, investigations on a *triple* solution of the problem. The explanation of this seeming contradiction is the following:—

It was pointed out by Oppolzer in the year 1882 (compare also his *Lehrbuch der Bahnbestimmung*, p. 308 ff.) that the equation (1) from which, in the method of Olbers, the distance of the comet is determined, sometimes may have *three* real positive roots. The comet Cruls 1882 gave him a direct instance of such a case. As he did not put the equation (1) in relation to the equation of Lagrange, he was unaware of the fact that, of these three solutions, only *one* was admissible. He was thus led to the somewhat paradoxical conclusion that the problem of determining an elliptical (or hyperbolical) orbit from three observations could sometimes lead to a double solution, but that the more simple problem concerning a parabolic orbit allowed of a triple solution! We have found that a contradiction between these two cases, in fact, does not take place.

2. The results of Oppolzer and Tscherny were obtained from the equation (1) of Euler. We are led to the same conclusions if the method of Laplace or that of Lagrange is taken as basis of discussion. Using the method of Laplace, I have treated the subject in the following way in my lectures of this year.

The fundamental equation in the problem of orbits is the equation of Lagrange, which, with the notation of my previous note, has the form

$$u = \lambda(1 - v^{-3}) \quad . \quad . \quad . \quad . \quad (2)$$

This equation holds for all orbits (for all conic sections). Between u and v there exists the relation

$$v^2 = 1 + u^2 + 2nu \quad . \quad . \quad . \quad . \quad (3)$$

If the orbit is supposed to be a parabola, the *vis viva* integral gives a new relation

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2 = \frac{2}{r} \quad . \quad . \quad . \quad (4)$$

which, in terms of the variables u and v, takes the form

$$au^2 + 2bu + 1 = 2/v \quad . \quad . \quad . \quad . \quad (5)$$

a and b being constants depending on the observed values of the coordinates, satisfying the inequality

$$a > b^2 \quad . \quad . \quad . \quad . \quad . \quad (6)$$

Eliminating v with the help of (3), the equation (5) takes the form

$$(u^2 + 2nu + 1)(au^2 + 2bu + 1)^2 = 4 \quad . \quad . \quad (5a)$$

This equation is of the sixth degree and has the constant term negative ($= -3$). It is identical with the equation deduced by Oppolzer from the equation (1) of Euler, as will be shown below. It has one or, eventually, three positive roots, the latter giving the "threefold" solution of the comet-problem.

Of these three roots, however, only that or those are admissible that simultaneously satisfy the equation (2) of Lagrange. We may deduce from (2) and (5) another equation more suitable for the discussion of the common roots.

We can derive an equation of the third degree in the following way. From (2) we get

$$1/v^3 = 1 - u/\lambda.$$

Hence we have

$$1/v = v^2/v^3 = (u^2 + 2nu + 1)(1 - u/\lambda),$$

so that the equation (5) may be written

$$au^2 + 2bn + 1 = 2(u^2 + 2nu + 1)(1 - u/\lambda) \quad . \quad . \quad (7a)$$

This equation of the third degree has first been given by Cauchy (1846). It may sometimes be useful in the numerical calculation of the root, but for our present purpose another equation of the sixth degree is preferable.

Substituting the value (2) of u in (5), we get, after some reductions,

$$(a\lambda^2 + 2b\lambda + 1)v^6 - 2v^5 - 2(a\lambda^2 + b\lambda)v^3 + a\lambda^2 = 0 \quad . \quad (7b)$$

Here the coefficient of v^6 is, according to the inequality (6), necessarily positive. Hence, from the theorem of Cartesius, it follows that this equation has, at most, two positive roots. The last term being positive, it must have exactly two such roots. The value of these roots may be more precisely determined. Taking the left-hand member of $(7b) = f(v)$, we have in fact

$$f(0) = a\lambda^2, \text{ positive.}$$
$$f(1) = -1.$$
$$f(\infty) = \text{positive.}$$

Hence there is one positive root less than unity and one root greater than unity.

It follows, however, from the equation (2) of Lagrange, that for λ positive we must have $v > 1$, and for λ negative we must have $v < 1$.

Hence the determination of a parabolic orbit from three complete observations can only take place in a single way.

We further, from this reasoning, draw the conclusion that whenever the equation (5*a*)—or the corresponding equation of Oppolzer—happens to have three real positive roots, only one of these roots is consistent with the equation of Lagrange, so that only one of the roots gives a solution of the problem. It can never occur that a triple solution appears in determining a parabolic orbit of a comet.

3. Oppolzer deduces his equation of the sixth degree from the equation (1) of Euler, developing the left member into powers of s and retaining only the first term. He thus gets

$$s = 2k(t_3 - t_1)/(r_1 + r_2)^{\frac{1}{2}}.$$

Putting

$$t_2 = k(t_3 - t_1)$$
$$2r_2 = r_1 + r_3$$

we hence have

$$s^2 = 2t_2^2/r_2 \quad . \quad . \quad . \quad . \quad . \quad (8)$$

Remembering now that s^2 can be expressed as a quadratic function of the distance of the comet from the Earth, we get immediately the equation of Oppolzer of the sixth degree.

It is, however, easily shown that the equation (8) approximately coincides with the *vis viva* integral (4). Denoting the differential of the arc by ds, this equation has the form

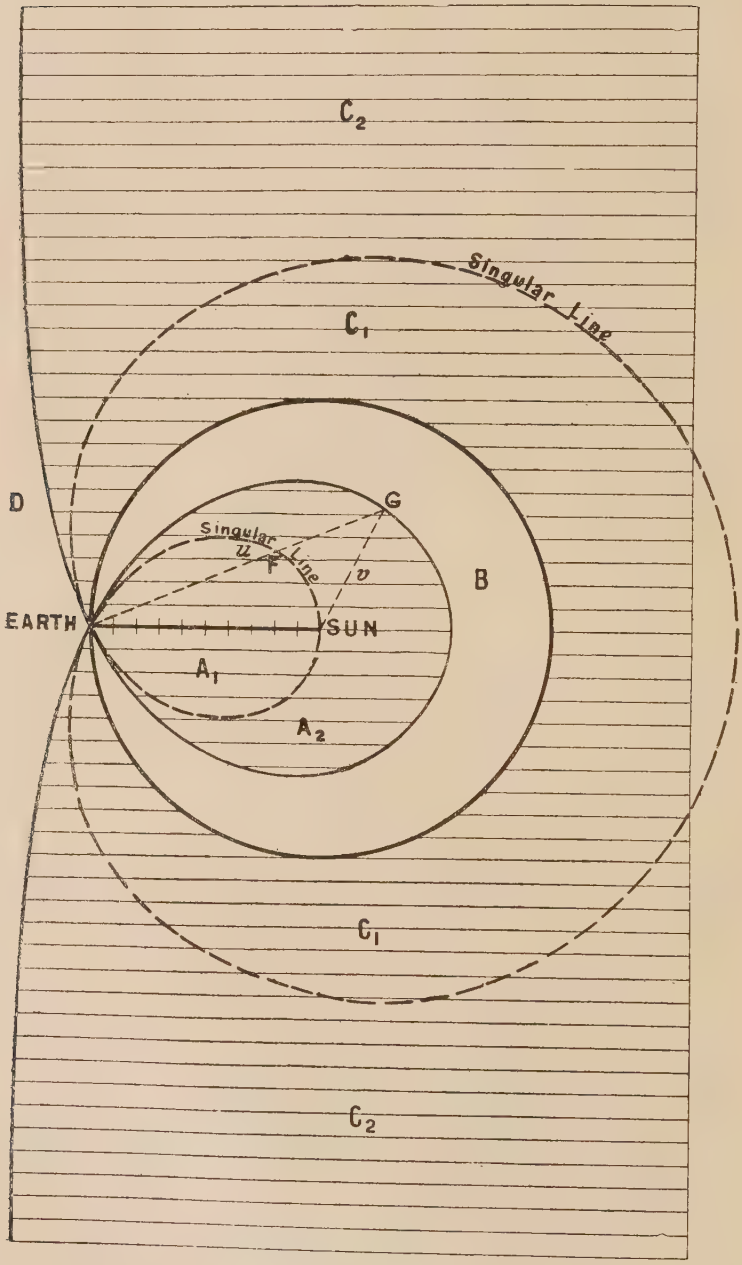
$$ds = \sqrt{(2/r)}dt,$$

which equation, integrated between $t = t_1$ and $t = t_3$, supposing r constant in this interval, gives a relation identical with the equation (8) of Oppolzer.

4. How is it, then, that, even in the determination of a parabolic orbit, multiple solutions sometimes occur, though, mathematically speaking, only a single solution is possible?

As to the two comets (Cruls 1882 and Innes, 1910), regarding which the computers have been led to a multiple solution, there is no doubt that the explanation is that the computation has been performed with the method of Olbers without taking into account the equation of Lagrange. Had this been done, the computers would have found that of the three roots of the equation of Euler only one is really admissible.

But if, in the regions A and C, the equation of Lagrange gives rise to conic sections that are both *nearly* parabolic, it may be, on account of the uncertainty in the observations, difficult to decide


C2
Singular Line
C1
D
G
Singular Line
u
v
B
EARTH
SUN
A1
A2
C1
C2

which of the two solutions most nearly coincides with the parabolic hypothesis. A fourth observation is then wanted for an unequivocal determination of the orbit.

5. I am able now to complete my previous note on the regions of double solutions in an important point. There exists, namely, a line, or more exactly expressed, a surface of revolution, that enjoys some remarkable properties. I call this line *the singular line* (surface) in the problem of orbits. Its equation in bi-polar coordinates is

$$u^2 = 1 - 5v^2/3 + 2v^5/3.$$

Its form is seen from the accompanying diagram.

By this singular line the regions of double solutions A and C are separated into two dominions A_1 and A_2, and C_1 and C_2. Now, if the true comet is in one of those dominions, then the "false comet" (corresponding to the other root of the equation of Lagrange) is situated in the other dominion; if the true comet is in A_1, then the false comet is in A_2, and so on. More precisely, taking a line E F G through the Earth, meeting the singular line in F and the border of the region A in G, if the true comet is in G, then the false comet coincides with the Earth. As the true comet proceeds, in the dominion A_2, from G towards the point F, the false comet proceeds, in the dominion A_1, from the Earth towards the same point F. At F the true comet and false comet coincide.

It is seen from this that the singular surface gives the position of those points in space where the two real positive roots of the equation of Lagrange coincide. With this fact is closely allied another property of this line which is the real justification for its characterisation as the singular line in the problem of orbits. Suppose three geocentric observed positions of a comet or a planet to be given. Then the elements of the (unknown) orbit can be developed, for moderate intervals of time, into convergent series proceeding according to the powers of the intervals of time between the three observations. The coefficients in these series are known functions of the observed geocentric coordinates. I have determined the first terms in these expansions. These series can be formed, and are convergent, *provided that the planet from the first to the third observation*—this interval being supposed to be of moderate length—*does not cross the singular line* (surface). On the contrary, if this is the case, the elements cannot be developed in powers of the intervals of time between the observations. Expansions of another form are then to be employed.

Lund:
1911 *March* 6.

On the Problem of Distribution in Globular Star Clusters.

By H. C. Plummer, M.A. (Plate 8.)

1. Among the most remarkable objects in the sky must be reckoned the globular clusters. They suggest a number of interesting problems, the most elementary and fundamental of which is concerned with the statistical arrangement of the stars they contain. Simple counts form the basis of this study, and these can be made very easily when we possess good photographs. The close aggregation of the stars, however, demands high resolving power of the telescope, and the technical difficulty in photographing these objects successfully is very considerable. Hence the structure of stellar clusters has only received serious study in recent years. The first important investigation of the subject was made by Professor E. C. Pickering* about twenty years ago. He arrived at these two main conclusions:—

(1) The law of distribution is essentially the same for different clusters.
(2) The bright stars and the faint stars of a cluster obey the same law.

In search of the law of distribution of the stars in space he tested the forms $1 - r^2$ and $(1 - r)^n$, making the calculation for the latter with the values $n = 1$, 2, 3, and 4. The results were not very satisfactory. The same form of law was applied independently to the cluster M 13 by Mr. W. E. Plummer,† who gave the numerical results up to $n = 12$. The agreement with the observed numbers was not very close, nor could it be expected with a purely empirical law of this kind. The investigation merely served to show, since a high value of n was indicated, that the concentration of the stars in space increases very rapidly towards the centre of the cluster.

More recently Dr. v. Zeipel‡ has made an important contribution in a study of the cluster M 3. He has shown how the law of distribution in space may be deduced numerically from the observed law of distribution in the projection as we see it. Further, he has compared§ the distribution so determined for the clusters M 13 and ω Centauri with the densities to be expected in a spherical mass of gas in isothermal equilibrium. The result is to represent the central parts of the cluster satisfactorily, while in the outer parts the cluster is less dense than this theory requires.

It is proposed in this note to pursue the search for a physical basis on which the distribution of stars in clusters may be

* *Harvard Annals*, xxvi.
† *Mon. Not.*, lxv. p. 810.
‡ *Annales de l'Obs. de Paris*, xxv.
§ *C. R.*, cxliv. p. 361.

established. And first we examine the statistical relations between the density of the stars in space and the numbers obtained by different methods of counting.

2. Let R be the radius of the cluster, which for the present we may consider finite. Let three rectangular axes Ox, Oy, Oz be taken through the centre, Oz coinciding with the line of sight. We are concerned with three functions of the distance r from the centre, $\phi(r)$, $f(r)$, and $F(r)$. Of these $\phi(r)dv$ denotes the number of stars in a small volume dv; $f(r)da$ denotes the number of stars in a cylinder of small section da, the axis being at a distance r from the centre; and $F(r)dr$ denotes the number of stars lying between the planes $x=r$ and $x=r+dr$. Then

$$F(r) = \int\int \phi(\rho) dy dz$$

where

$$\rho^2 = r^2 + y^2 + z^2, \quad R^2 > r^2 + y^2 + z^2.$$

The boundary being given by the inequality, the integral becomes

$$F(r) = \int_r^R 2\pi\rho \,.\, \phi(\rho) d\rho \quad . \quad . \quad . \quad (1)$$

whence

$$\phi(r) = -F'(r)/2\pi r \quad . \quad . \quad . \quad . \quad (2)$$

Now $F(r)$ is the function which can be most easily determined from a photograph, namely, by simple counts of the stars in narrow parallel strips. Such counts can be directly compared with the result of applying an assumed law of distribution ϕ, or this law can be determined numerically by mechanical differentiation of F. The conditions necessary for complete accuracy are (1) that the counts shall cover the whole extent of the cluster, and (2) that the stars belonging to the background (or foreground) of the sky shall be excluded. If $\Sigma(r)$ is the number counted between the centre and the plane $x=r$, we have

$$F(r) = \Sigma'(r), \quad \Sigma(r) = \int_0^r F(r) dr.$$

3. It has been the more general practice to determine the function f which represents the distribution of the stars in the projection perpendicular to the line of sight. This is

$$f(r) = 2\int_0^{\sqrt{(R^2-r^2)}} \phi(\rho) dz$$

$$= 2\int_r^R \frac{\rho\phi(\rho)}{\sqrt{(\rho^2 - r^2)}} d\rho \,. \quad . \quad . \quad . \quad (3)$$

since $\rho^2 = r^2 + z^2$. Integrating by parts, since $\phi(R) = 0$, we have

$$f(r) = -2\int_r^R \sqrt{(\rho^2 - r^2)}\phi'(\rho) d\rho,$$

and therefore

$$f'(r) = 2r\int_r^R \frac{\phi'(\rho)}{\sqrt{(\rho^2 - r^2)}}d\rho$$
$$= 2r\int_0^{\sqrt{(R^2-r^2)}} \frac{\phi'(\rho)}{\rho}dz.$$

If we now integrate along the line $x = r_1$, $z = 0$, we obtain

$$\int_0^{\sqrt{(R^2-r_1^2)}} \frac{f'(r)}{r}dy = 2\int_0^{\sqrt{(R^2-r_1^2)}} dy \int_0^{\sqrt{(R^2-r^2)}} \frac{\phi'(\rho)}{\rho}dz$$

where $y^2 = r^2 - r_1^2$. Hence

$$\int_{r_1}^R \frac{f'(r)}{\sqrt{(r^2 - r_1^2)}}dr = \pi\int_{r_1}^R \frac{\phi'(\rho)}{\rho}.\rho d\rho$$
$$= -\pi\phi(r_1) . \quad . \quad . \quad . \quad (4)$$

or, writing ρ for r_1, and integrating by parts,

$$\phi(\rho) = \frac{1}{\pi}\int_\rho^R \sqrt{(r^2 - \rho^2)}\frac{d}{dr}\left[\frac{f'(r)}{r}\right]dr . \quad . \quad . \quad (5)$$

if we may assume $f'(R) = 0$.

If the totals are counted up to successive distances r from the centre, and these are represented by $\sigma(r)$,

$$f(r) = \sigma'(r)/2\pi r.$$

The formulæ (4) and (5), for which we have given a simple, direct proof, were deduced by v. Zeipel from a theorem by Abel. It would appear that the application of this method of determining the function ϕ must be rather laborious.

4. If we suppose that a globular cluster is the outcome of a primitive spherical nebula, it is reasonable to expect that the density of matter at different distances from the centre will be approximately preserved. It is natural, therefore, to look for guidance to possible conditions of equilibrium in a spherical mass of gas. Let ϕ, p, t be the density, pressure, and temperature at the distance r from the centre of the nebula. For convective equilibrium in a perfect gas we have

$$p = at\phi = b\phi^\gamma$$

where a, b are constants and γ is the ratio of the specific heats. Also, if m is the mass within the radius r,

$$\frac{dp}{dr} = -cm\phi/r^2 , \quad \frac{dm}{dr} = 4\pi r^2\phi .$$

Eliminating p and m,

$$\frac{d}{dr}\left(\frac{br^2}{\phi}\cdot\frac{d\phi^\gamma}{dr}\right)+4\pi cr^2\phi=0 \quad . \quad . \quad . \quad (6)$$

or, choosing the units to give the results in the simplest form, since we are not concerned with absolute values,

$$\frac{d}{dr}\left(r^2\cdot\frac{d\phi^{\gamma-1}}{dr}\right)+r^2\phi=0 \quad . \quad . \quad . \quad (7)$$

If we change the independent variable to $u=1/r$, this takes the well-known form

$$\frac{d^2\phi^{\gamma-1}}{du^2}+\frac{\phi}{u^4}=0 \quad . \quad . \quad . \quad . \quad (8)$$

On the other hand, we get the corresponding equations for isothermal equilibrium by putting $\gamma=1$ in (6) and obtain similarly

$$\frac{d}{dr}\left(r^2\cdot\frac{d\,.\log\phi}{dr}\right)+r^2\phi=0 \quad . \quad . \quad . \quad (9)$$

$$\frac{d^2(\log\phi)}{du^2}+\frac{\phi}{u^4}=0\,.$$

5. These equations have received much study.* The appropriate solutions cannot in general be expressed mathematically in finite terms, but the more interesting cases have been worked out in a numerical form. The corresponding distributions either have a finite boundary, *e.g.* when $\gamma=1{\cdot}4$ (diatomic gases), or they extend to infinity, as in the isothermal state given by (9). There are, however, two special cases which are known to be susceptible of solution in a simple form. The first solution was discovered by Ritter † for the case of $\gamma=2$, namely,

$$\phi(r)=\sin r/r \quad . \quad . \quad . \quad . \quad (10)$$

and the second by Schuster ‡ for $\gamma=1{\cdot}2$, namely,

$$\phi(r)=(1+r^2)^{-\frac{5}{2}} \quad . \quad . \quad . \quad . \quad (11)$$

except for a numerical factor which we can omit. The values of γ for simple gases are near $1\frac{2}{3}$ (*e.g.* mercury), $1{\cdot}4$ (*e.g.* hydrogen), and $1\frac{1}{3}$ (*e.g.* chlorine), in accordance with the kinetic theory. But values less than $1{\cdot}2$ have been found for compound gases (*e.g.* chloroform), while greater values than 2 have been deduced by Amagat § for carbon dioxide under great pressures at moderate temperatures, though no values so great seem to have been determined by direct experiment. On the whole we may well be content to employ, at least in the first instance, that indication of

* *Cf.* Lord Kelvin, *Proc. R.S.E.*, xxviii. p. 259; Emden, *Gaskugeln* (Leipzig, 1907).

† Wiedemann's *Annalen*, xi. p. 338.

‡ *B.A. Report*, 1883, p. 428.

§ *C.R.*, cxxi. p. 863.

theory which appears in the simplest form. This is expressed by (11), and we have

$$\phi(r) = N(1+r^2)^{-\frac{5}{2}},\ F(r) = \tfrac{2}{3}\pi N(1+r^2)^{-\frac{3}{2}},\ \Sigma(r) = \tfrac{2}{3}\pi N r(1+r^2)^{-\frac{1}{2}} \quad (12)$$

$$f(r) = 2N\int_r^\infty \frac{\rho d\rho}{(\rho^2-r^2)^{\frac{1}{2}}(1+\rho^2)^{\frac{5}{2}}} = \frac{2N}{(1+r^2)^2}\int_0^\infty \frac{dq}{(1+q^2)^{\frac{5}{2}}}$$

where

$$q^2(1+r^2) = \rho^2 - r^2.$$

The definite integral is equal to $\frac{2}{3}$, and we have

$$f(r) = \tfrac{4}{3}N(1+r^2)^{-2}, \quad \sigma(r) = \tfrac{4}{3}\pi N r^2(1+r^2)^{-1} \quad . \quad (13)$$

6. As our first example we will take the counts of the ω Centauri cluster made by Professor S. I. Bailey.* The results of a comparison with the formula (12) are shown in the following Table I. :—

TABLE I.

ω Centauri.

To r =	S.	N.	W.	E.	Means.	Sums.	C.	O−C.	ΔC.	Δ(O−C).
1′5	777	781	681	672	728	728	727	+ 1	727	+ 1
3·0	689	673	637	632	658	1386	1371	+15	644	+14
4·5	524	538	470	506	510	1896	1887	+ 9	516	− 6
6·0	392	348	335	476	388	2284	2277	+ 7	390	− 2
7·5	248	249	280	336	278	2562	2564	− 2	287	− 9
9·0	197	156	201	253	202	2764	2773	− 9	209	− 7
10·5	132	140	146	185	151	2915	2927	−12	154	− 3
12·0	128	119	110	141	125	3040	3042	− 2	115	+10
13·5	92	64	78	104	85	3125	3129	− 4	87	− 2
15·0	75	67	63	83	72	3197	3196	+ 1	67	+ 5
∞							3540			
								+33		+30
								−29		−29

Here the four columns following the first contain the numbers of stars in complete strips, the edges of which are at the given distances from the centre of the cluster. The means are given in column 6; incidentally, the use of means is easily seen to reduce the effect of an error in the assumed position of the centre. Column 7 contains the derived sums from the centre to the distances given, the last being half the total number for the cluster. These are compared in the next two columns with the formula

$$\Sigma(r) = 3540r/\sqrt{(1+r^2)}$$

* *Astronomy and Astro-Physics*, xii. p. 689; *Harvard Annals*, xxvi. p. 213.

the unit of r being chosen at 7'·14. If the counts were extended to infinity we should thus expect to find 7080 stars, as against 6389 counted. In the last column but one the computed sums are again split up according to the successive strips, and the last column gives the residuals when the comparison is made with the means in column 6. When the divergence of the separate numbers in columns 2–5 is considered, it is clear that the counts are at least as well represented by the formula we have chosen as they are consistent with the fundamental hypothesis that we are dealing with a truly spherical distribution.

One qualification must be made. We have made no allowance for the average density of the sky apart from the cluster. It is exceedingly difficult to do this with any certainty. According to our present ideas the cluster has no finite boundary. Hence an estimate based on counts in the vicinity will be too high, because we shall include stars which belong to the cluster. On the other hand, counts made at a considerable distance may not represent the density of the actual region. For the present it is enough to say that the neglect of this correction can scarcely invalidate the excellent agreement which has been found between the counts and the formula.

7. Before going further in testing the law indicated by (11), it will be well to examine the suitability of the other simple law, represented by (10). This gives

$$F(r) = \int_r^R \sin\rho \,.\, d\rho = 1 + \cos r$$

$$\Sigma(r) = \int_0^r F(r)dr = r + \sin r$$

when a constant factor, $2\pi N$, is ignored. Here the distribution has a finite boundary, $R = \pi$. If we assume that the counts extend to this boundary and are complete within it, we have the results shown in Table II. when we multiply $\Sigma(r)$ by 1018 to give the proper total.

TABLE II.

r.		Sums.	C.	O – C.	Means.	ΔC.	Δ(O – C).
1'·5 =	18°	728	634	+ 94	728	634	+ 94
3·0	36	1386	1238	+ 148	658	604	+ 54
4·5	54	1896	1784	+ 112	510	546	– 36
6·0	72	2284	2248	+ 36	388	464	– 76
7·5	90	2562	2617	– 55	278	369	– 91
9·0	108	2764	2887	– 123	202	270	– 68
10·5	126	2915	3062	– 147	151	175	– 24
12·0	144	3040	3158	– 118	125	96	+ 29
13·5	162	3125	3193	– 68	85	35	+ 50
15·0	180	3197	3199	– 2	72	6	+ 66
				+ 390			+ 293
				– 513			– 295

Here the first two columns contain the distance from the centre in actual angular measure and in the angular unit appropriate to the formula. Columns 3 and 6, which are identical with columns 7 and 6 in Table I., give the totals in strips up to these distances, and their differences. Column 4 gives the computed totals, with the differences in column 7. The results of the comparison are given in columns 5 and 8. There can be no doubt that the formula fails to represent the counts. It may be that a better appearance might be obtained by altering the scale value in r and adjusting the density by a different factor. Also, as before, the correction for general sky density has been omitted. But it does not appear that a good agreement could be obtained by such means. The evidence suggests strongly that in comparison with the law $(1+r^2)^{-\frac{3}{2}}$ the law $\sin r/r$ may be set aside for our purpose.

8. The curves which show the number of stars in the projection at different distances from the centre can be made directly comparable by reducing the central density to unity and choosing an appropriate scale value in distance. This has been done by Professor Pickering for the clusters ω Centauri, 47 Tucanæ (bright and faint stars separately), and M 13 (for which the material was derived from Professor Scheiner's measures). The actual densities (central or mean) and scale values cannot be insisted on as material, since the former is a function of the optical power and time of exposure, and the latter depends on the distance of the cluster. In fact, the counts are never complete and the parallaxes are unknown. Through the points which he had thus reduced to a common system Professor Pickering drew a smooth curve to represent the general law. This curve we now compare with the law represented by (13),

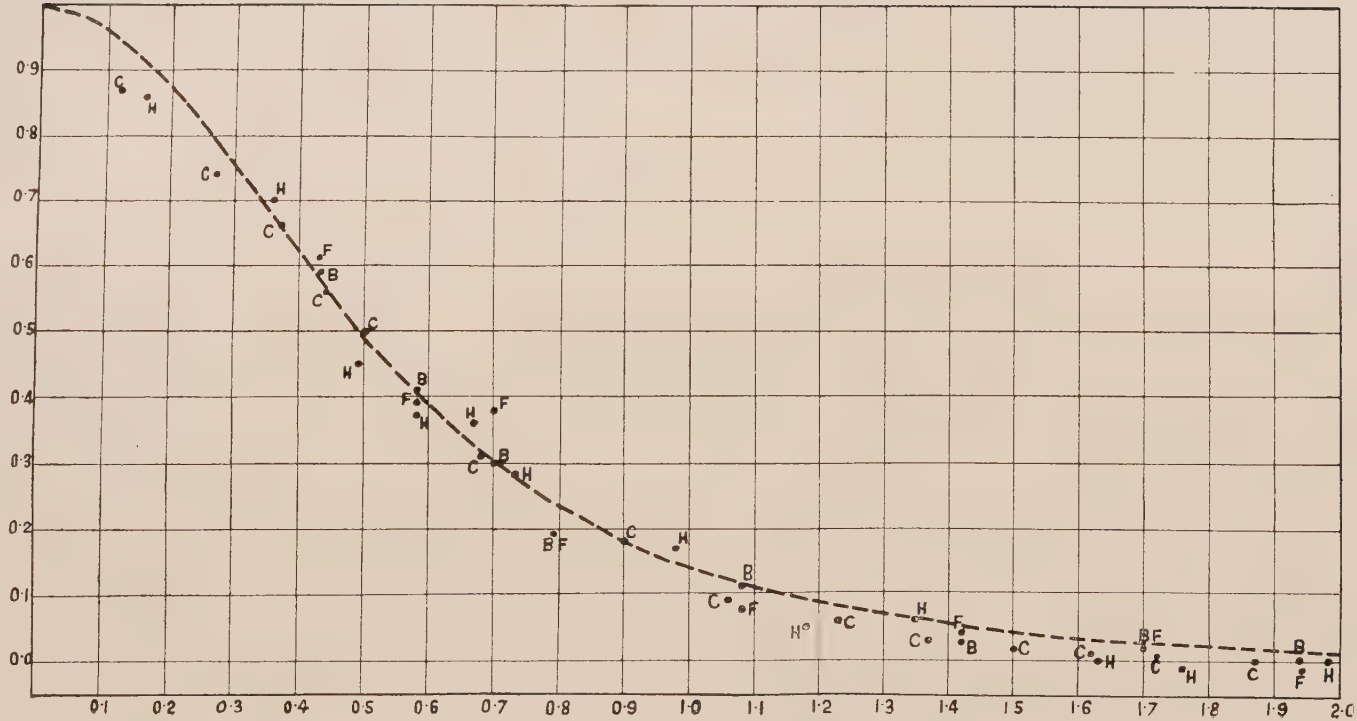
$$y = (1+r^2)^{-2}$$

but making $r = 1{\cdot}29\ x$. The result is shown in Table III.

Table III.

x.	Curve.	Formula.	F−C.	x.	Curve.	Formula.	F−C.
0·0	1·000	1·000	·000	1·0	0·132	0·141	+·009
0·1	0·900	0·968	+·068	1·1	·098	·110	+·012
0·2	·800	·879	+·079	1·2	·071	·087	+·016
0·3	·700	·756	+·056	1·3	·050	·069	+·019
0·4	·600	·624	+·024	1·4	·034	·055	+·021
0·5	·500	·499	−·001	1·5	·022	·044	+·022
0·6	·402	·391	−·011	1·6	·013	·036	+·023
0·7	·311	·303	−·008	1·7	·007	·030	+·023
0·8	·235	·235	·000	1·8	·003	·024	+·021
0·9	·176	·181	+·005	1·9	·001	·020	+·019
1·0	·132	·141	+·009	2·0	·000	·017	+·017

CURVE OF APPARENT DENSITY IN CLUSTERS.



C=ω Centauri, B=47 Tucanæ (bright stars), F=47 Tucanæ (faint stars), H=M 13.

It will be noticed that the values given by the formula are on the average ·020 in excess of the values given by the curve. This is due, it may fairly be contended, to a too liberal allowance being made for the general sky average in the determination of the Harvard curve. That the stars of a cluster die out at a particular distance is confessedly an arbitrary assumption. It is right that the law should be compared not simply with the interpolated curve but also with the actual points on which that curve is based. This is done in the figure (Plate 8), in which the points are reproduced from the diagram given by Professor Pickering. We see that the agreement of the points with the curve now proposed is good, and that it would be even better if the points were all raised by the amount ·020.

9. We have thus found that the evidence drawn from the Harvard material lends support to the idea that the distribution of stars in clusters resembles the distribution of density in a gravitating sphere of a particular kind of gas to which Schuster's solution applies. It would seem, at all events, that this simple form of law deserves to be borne in mind when further statistical results become available. At present the material for discussion is scanty. But we have v. Zeipel's measures of the cluster M 3, to which reference has already been made, and the examination of this case will be found useful, since it will warn us not to attribute excessive generality or precision to the preceding results. Of the three plates used in forming the complete catalogue, the one with the longest exposure (2^h) alone is used in making the counts by strips parallel to the axis of y. The centre of the cluster is assumed at $x = +2'{\cdot}1$, and means of the numbers found on opposite sides of this centre are used. It is noted (p. 17) that "une trentaine d'étoiles" in the centre of the cluster could not be included among the measures owing to the density of the plate; on this account we add 15 stars to the central strip. A further correction is made by subtracting 5 stars for each strip 1′ wide to allow for the average density of the sky apart from the cluster. We now have the numbers in Table IV.

Table IV.

r.	O.	$\Sigma(r)$.	Diff.	C.	Δ.	r.	O.	$\Sigma(r)$.	Diff.	C.	Δ.
0′·5	135	148	148	137	137	6′	549	534	16	533	11
1·0	253	263	115	251	114	7	559	539	5	541	8
1·5	337	345	82	337	86	8	573	548	9	545	4
2·0	388	393	48	397	60	9	582	552	4	549	4
2·5	432	435	42	438	41	10	592	557	5	551	2
3·0	461	461	26	468	30	15	630	570	13	557	6
4·0	501	496	35	502	34	20	664	579	9	559	2
5·0	528	518	22	522	20	25	691	581	2	560	1

Here the column headed O contains the means of counts to

the given distances on both sides of the centre, the column headed $\Sigma(r)$ the corresponding numbers when the corrections just explained have been applied, and the next column the differences of the preceding, *i.e.* the number of stars belonging to the cluster in each strip. With these last two columns are to be compared the columns headed C and Δ respectively, in which the numbers calculated from the formula

$$\Sigma(r) = 562r(1 + r^2)^{-\frac{1}{2}}$$

with 2′ as the unit of r, and their differences are given. The general agreement is fairly good, and if no other tests were available, there would be little reason to be dissatisfied with this representation of the distribution of the stars.

10. Dr. v. Zeipel has given (p. 28) the counts of the stars lying between circles drawn at regular distances around the centre of the cluster. These counts, corrected for the average density of the sky, determine $\sigma(r)$ for different values of r, and we must compare them with formula (13), which in this case we make

$$\sigma(r) = 1160r^2(1 + r^2)^{-1}$$

with 2′ as the unit of r as before. This gives the numbers in the column C of Table V.

Table V.

r.	$\sigma(r)$.	Diff.	C.	Δ.	R.	r.	$\sigma(r)$.	Diff.	C.	Δ.	R.
0′·5	103	103	68	68	213	5′	959	59	1000	72	280
1′·0	326	223	232	164	212	6	1009	50	1044	44	283
1′·5	499	173	418	186	238	7	1045	36	1073	29	278
2′·0	634	135	580	162	244	8	1075	30	1092	19	279
2′·5	747	113	706	126	253	9	1091	16	1105	13	279
3′·0	819	72	803	97	259	10	1115	24	1116	11	278
4′·0	900	81	928	125	273	13	1144	29	1133	17	278

The comparison of the fifth column (headed Δ) with the third column, which contains the corrected counts in rings about the centre, reveals a want of agreement which can only be attributed to the inability of our formulæ to represent the true structure of the cluster. In particular, the central part of the cluster is much denser than our theory allows, and there is one simple test which shows this very clearly. If we compare the formulæ (12) and (13), we have simply

$$\{\Sigma(r)\}^2/\sigma(r) = \tfrac{1}{3}\pi\mathrm{N} = \tfrac{1}{2}\Sigma(\infty).$$

It is true that in this case we have deduced $\Sigma(r)$ from one plate only, while $\sigma(r)$ contains a few stars in addition which occur on two plates of shorter exposure. But this will not invalidate the

conclusion that the defined ratio should be constant for all values of r. This test is the more convenient because, unlike the previous comparisons, it is independent of the assumed unit to which r is referred. Accordingly, the ratio deduced from the numbers shown in Tables IV. and V. is given under R in Table V. The numbers show a systematic run which it would be impossible to alter greatly by any reasonable change in the assumed density of the stars not belonging to the cluster. The direction of the run proves unmistakably that the concentration of the cluster towards the centre is greater than is possible according to the law which is the subject of examination.

11. The same point is brought out very clearly in another way. We have used $2'$ as the unit of r and 1160 as the numerical factor in $\sigma(r)$. This latter is $\frac{4}{3}\pi N$ according to (13), whence $N = 277$. Thus

$$\phi(r) = 277(1 + r^2)^{-\frac{5}{2}}/(2')^3 = 34{\cdot}6(1 + r^2)^{-\frac{5}{2}}/(1')^3.$$

Now, Dr. v. Zeipel has calculated numerically by means of (5) the values of $\phi(r)$ corresponding to the numbers found for $f(r)$. These we can now compare with the above formula, and the result is shown in Table VI., where the numbers calculated from the formula are given under the heading C.

TABLE VI.

$4r$.	$\phi(r)$.	C.	$4r$.	$\phi(r)$.	C.	$4r$.	$\phi(r)$.	C.
1′	58·5	33·3	8′	4·83	6·11	19′	0·19	0·30
2	56·6	29·8	9	4·18	4·52	21	0·17	0·20
3	48·0	24·9	10	3·34	3·30	23	0·14	0·13
4	31·6	19·8	11	1·84	2·45	25	0·10	0·09
5	16·2	15·2	13	0·96	1·37	27	0·06	0·06
6	8·81	11·3	15	0·41	0·80	29	0·06	0·05
7	6·26	8·34	17	0·21	0·49	31	0·06	0·03

The deficiency in the calculated density of the cluster within 1′ of the centre is very marked. Beyond this distance the agreement improves. As the numbers $\phi(r)$ themselves are not free from irregularities which become very evident when $r\phi(r)$ is tabulated and differenced, and which could with difficulty be reconciled with any ordinary function, the representation of the density in the outer parts of the cluster is fairly satisfactory, but the discrepancy in the central density is, at first sight, serious.

This difficulty, however, by no means compels us to abandon the general physical basis on which we have proceeded. It has been mentioned in § 1 that Dr. v. Zeipel found that the distribution in the central parts of this cluster is satisfactorily represented by the density of a sphere of gas in isothermal equilibrium. This

law failed to correspond with the density observed in the outer parts, while the opposite state of things presents itself when we seek to apply a particular law of convective equilibrium. Hence we are led to combine both solutions and to conclude that the cluster is the outcome of a nebula in convective equilibrium outside a central core in isothermal equilibrium. There is nothing strange in such a distribution, for Ritter and others who have investigated the subject of equilibrium in gaseous spheres have contemplated the possibility of such isothermal-adiabatic phases. This combination will occur when convective currents transfer portions of the gas in the outer regions too quickly to allow an equilibrium of temperature to be established, while in the interior the greater density of the gas and the comparative freedom from disturbances make conduction operative and consequently the isothermal condition possible. Thus the facts obtained from the cluster M 3 can be explained without doing violence to the general line of thought which we have followed.

12. The contents of this paper are here summarised :—

§ 1. Introductory.

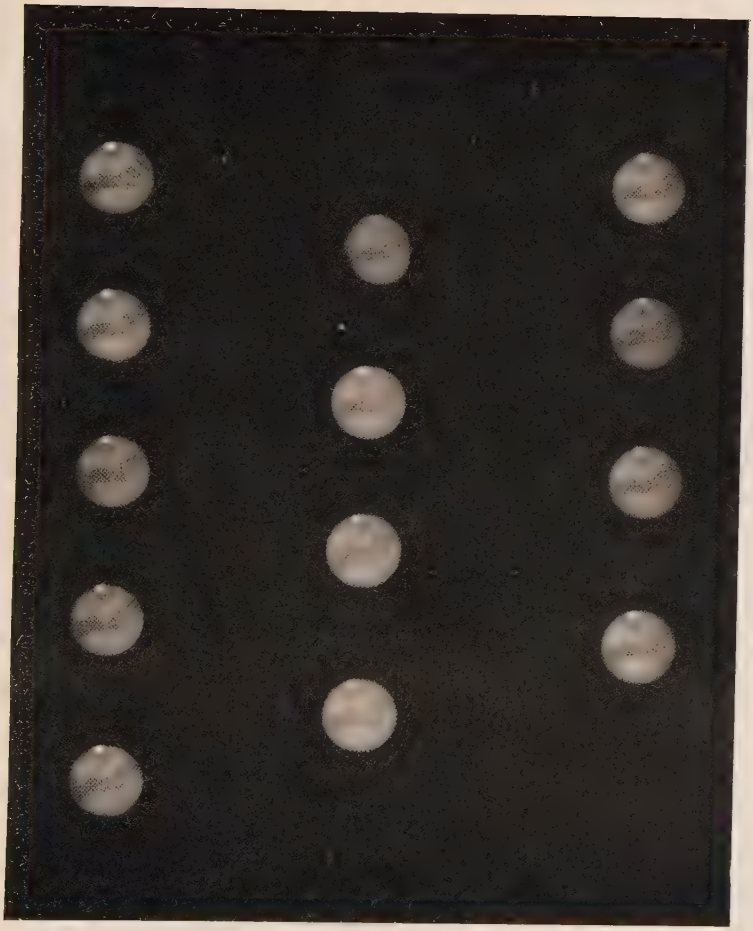
§§ 2, 3. Examination of the relations between the distribution of the stars of a cluster in space and in projection, and of the methods of making counts.

§§ 4, 5. The conditions of equilibrium of a spherical mass of gas with the two simplest forms of solution.

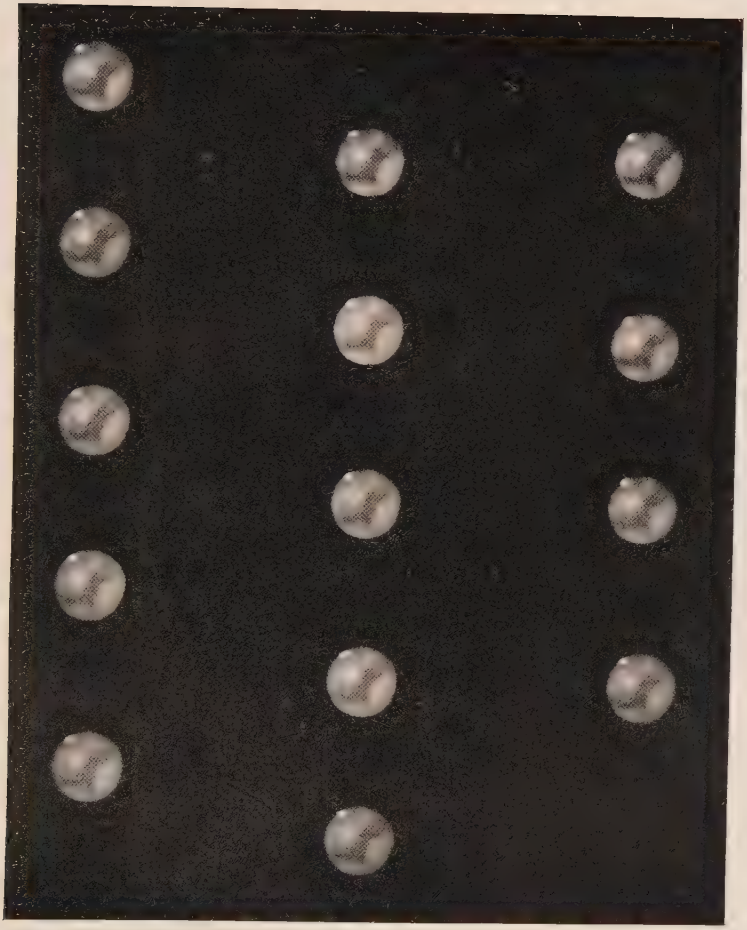
§§ 6, 7. The numbers counted in the ω Centauri cluster accord with one law $[N(1+r^2)^{-\frac{5}{2}}]$ and not with the other $[N \sin r/r]$.

§ 8. The same law represents well the apparent (projected) distribution found by Professor Pickering for the clusters ω Centauri, 47 Tucanæ (bright and faint stars), and M 13.

§§ 9–11. The cluster M 3 is compared with the same law and found to be marked by a higher degree of condensation towards the centre than can be reconciled with the law. It is suggested that this can be explained by supposing that the original nebular matrix possessed a central core in isothermal equilibrium, with an outer envelope in convective equilibrium.



Mars, 1909 September 24. 16h. 55m. G.M.T.—E. E. Barnard.



Mars, 1909 September 28. 16h. 46m. G.M.T.—E. E. Barnard.

Photographs of the Planet Mars, etc. By E. E. Barnard.
(Plates 9, 10.)

The accompanying glass positives, which are sent to the Royal Astronomical Society through the courtesy of the Yerkes Observatory, were made by the writer from the original negatives which were taken by him of the directly enlarged image of the planet Mars with the 40-inch refractor in 1909.

A full account of the making of these pictures will be printed later in the *Astrophysical Journal*. A brief statement of the work, however, is necessary here in connection with the present photographs.

The images were directly enlarged with a Brashear enlarging lens at the focus of the great telescope, and have received a slight subsequent enlargement. A yellow colour-filter made by Mr. R. J. Wallace was used with a Cramer Instantaneous Iso plate, with an exposure of three or four seconds.

Better conditions are required for successful work in this direction than for visual observations. One can do much visually under conditions where the best definition is only momentary; but for these enlarged photographs any break in the definition for even a single second during the exposure means injury or total ruin to the image. In all the exposures, though of only a few seconds' duration, it was necessary to guide the telescope to keep the image stationary. This was done by bisecting the polar cap by cross-wires (spider threads) in the focus of the long guiding finder ($61\frac{1}{2}$ feet focus) of the 40-inch telescope. Fairly satisfactory conditions were obtained on one night only, that of 1909 September 28, but several photographs made on September 24, which are less satisfactory, are also sent, as they may be of interest.

List of the Glass Positives ($6\frac{1}{2} \times 8\frac{1}{2}$ inches).

	h	m		
1909 September 24	16	55	G.M.T.	Region of the Fastigium Aryn and Margaritifer Sinus.
	17	15		
	18	17		
	19	10		
1909 September 28	15	24		Region of the Syrtis Major.
	15	44		
	16	24		
	16	46		

Accompanying these photographs of the planet Mars are two $6\frac{1}{2} \times 8\frac{1}{2}$-inch glass negatives. These are—

The Great Nebula of ν Scorpii.—In the negative form the plate more readily shows the effect of apparent partial obscuration of the light of the stars that seem to be behind the nebula. See *Astrophysical Journal*, xxxi., for January 1910.

Remarkable region north of θ Ophiuchi.—This is one of the most remarkable regions that I know of in the sky. It exhibits a wonderful amount of small and singularly sharp detail, such as could scarcely occur in a background of stars alone. It therefore suggests the presence of a nebulous background. See *Astrophysical Journal*, ix., for March 1899.

The Yerkes Observatory:
1911 *February* 15.

Bright Clock Stars—Night versus *Day.*

By W. G. Thackeray.

(*Communicated by the Astronomer Royal.*)

The question has been often asked as to what is the difference between the transit of a star observed in the daytime and the same star observed at night, and therefore whether the stars observed at the beginning and end of a twelve-hour group of clock stars are strictly comparable. There are several sources of possible error:

1. Diurnal variations in the level or azimuth.
2. Difference of magnitude equation.
3. Variation between the daylight and artificial illumination of the spider webs.
4. A faulty value of the aberration constant.

The difficulty is to find material which shall not be spread over too long a period of time, yet shall suffice to determine small quantities. The twelve stars selected for the discussion were chosen as sufficient to detect any possible error depending on R.A.; in fact, when one says chosen, it is a case of Hobson's choice, for among the limited number of stars that are available, some get few or no observations both by day and night.

The observations used extend from the year 1853 to 1906, and they have been grouped in periods (except for the first) corresponding with the Greenwich Catalogues.

The first period, 1860, depends on results spread over the years 1853–1867, which were taken from the new results depending on 12-hour groups (*M.N.*, lxxi. p. 182), and as Procyon was not used in this discussion there are no results for this period for that star. The results given under the headings of the other dates depend on the difference between the mean places given in the ledgers for the dates corresponding to the period when each star was observed by day or by night. As a general rule the observations were well within their respective zones, so that there were few cases where doubt arose as to whether they might have been observed by daylight or artificial light. The first table gives the differences of R.A. taken Night minus Day arranged in order of R.A. The second table gives the same differences arranged in order of N.P.D. The third table gives the number of observations made in the day and at night on which the quantities Night minus Day depend.

TABLE I.

Name of Star.	Approx. R.A.	Approx. N.P.D.	Night minus Day. 1860.	1872.	1880.	1890.	1900.	Mean.
	h m	°	s	s	s	s	s	s
β Androm.	1 4	54·9		+ ·019	– ·013	– ·008	– ·027	– ·011
Aldebaran	4 30	73·7	– ·024	– ·021	– ·041	– ·022	– ·024	– ·026
Rigel	5 10	98·3	– ·060	– ·038	– ·035	– ·021	– ·047	– ·040
α Orionis	5 50	82·6	– ·006	– ·009	– ·050	– ·022	– ·035	– ·025
Castor	7 28	57·9	+ ·003	+ ·048	·000	– ·027	·000	+ ·004
Procyon	7 34	84·5		– ·003	– ·037	– ·017	– ·028	– ·020
Pollux	7 39	61·7	– ·035	+ ·027	– ·027	– ·024	– ·018	– ·020
Regulus	10 3	77·5	– ·013	+ ·012	– ·015	– ·032	– ·028	– ·015
Spica	13 20	100·6	– ·036	– ·012	– ·016	– ·012	– ·027	– ·020
Arcturus	14 11	70·3	– ·043	– ·017	– ·020	– ·008	– ·013	– ·024
α Lyræ	18 34	51·3	+ ·017	– ·017	– ·005	– ·005	– ·009	– ·004
α Aquilæ	19 46	81·4	– ·015	– ·054	– ·059	– ·037	– ·030	– ·037
		Mean	– ·022	– ·008	– ·024	– ·019	– ·024	– ·020

TABLE II.

Name of Star.	Approx. R.A.	Approx. N.P.D.	Night minus Day.
	h m	°	s
α Lyræ	18 34	51·3	– ·004
β Androm.	1 4	54·9	– ·011
Castor	7 28	57·9	+ ·004
Pollux	7 39	61·7	– ·020
Arcturus	14 11	70·3	– ·024
Aldebaran	4 30	73·7	– ·026
Regulus	10 3	77·6	– ·015
α Aquilæ	19 46	81·4	– ·037
α Orionis	5 50	82·6	– ·025
Procyon	7 34	84·5	– ·020
Rigel	5 10	98·3	– ·040
Spica	13 20	100·6	– ·020
		Mean	– ·020

TABLE III.

Number of Observations.

	1860.		1872.		1880		1890.		1900.	
	Day.	Night.	Day.	Night.	Day.	Night.	Day.	Night.	Day.	Night.
β Androm.	...	...	38	23	44	44	104	79	65	80
Aldebaran	46	37	58	46	27	35	90	60	100	58
Rigel	38	42	43	30	34	31	58	36	74	64
α Orionis	51	28	69	20	32	25	89	43	107	63
Castor	28	8	36	21	36	33	83	40	94	51
Procyon	...	...	78	56	59	23	118	38	164	56
Pollux	53	20	82	33	62	21	114	55	163	71
Regulus	51	17	63	59	39	22	90	52	91	58
Spica	66	50	154	98	83	69	89	40	137	59
Arcturus	72	18	106	58	108	16	190	56	217	54
α Lyræ	44	23	83	118	87	103	143	118	166	83
α Aquilæ	57	22	58	53	48	37	82	93	86	113
Totals	506	265	868	615	659	459	1250	710	1464	810

Seeing the small quantity representing systematic error and the large accidental errors involved in the observations, it seems reasonable to assume that the mean difference $-0^s.020$ has been constant throughout the life of the transit circle as the average difference between a night and a day observation, though when arranged in order of N.P.D. the stars above 60° N.P.D. appear to give consistently smaller results. Any effect of variation in the level error could be but small, as the maximum change takes place about 6 o'clock.

The quantity $-0^s.020$ implies that the observations of R.A. made in the day-time are later than those made at night-time by $0^s.020$, which may be accounted for as a shift of the spider webs of the nature of a collimation error. The illumination of the wires of the transit-circle up till 1908 was regulated by an annulus, but since that date a system of central illumination has been instituted. A comparison made between the two systems of illumination, by taking two clock stars each night under the old system of illumination, has shown that the clock error derived from transits with the annular system is $0^s.022$ later than the clock error derived from transits with the new central system. As the central system is of much the same character as daylight illumination, it appears that the difference in R.A. between the stars observed in the day-time can be explained by a shift in the spider webs due to the system of the illumination.

There would thus be little or nothing left over for magnitude equation. What one would expect would be that, the action of daylight on the bright stars being to diminish their brilliancy, a transit taken by day would tend to be the same as that of a fainter star, namely later, or the R.A. would be bigger; but on the other hand the value of aberration used in the N.A. for these years is probably $0''.06$ small, and Professor Turner has pointed out (*Observatory*, vol. xxiii. pp. 70, 71), that the adoption of the N.A. star places without second differences is equivalent to taking too small an aberration value by $0''.05$. The resulting correction for aberration would be in the direction of making the daylight transits too early; if these two corrections absorb one another, it can only be that the magnitude equation of bright stars observed in the day as compared with the same stars observed at night is very small, and can have no injurious effect on 12-hour groups of clock stars.

From Table III., giving the number of observations for day and night respectively, it will be seen that nearly two-thirds of the observations of the bright stars are made by day, and therefore the places of the bright stars in the Greenwich Catalogues require on an average a correction of $-0^s.020 \times \frac{2}{3}$ or $-0^s.013$ to reduce them to the system of night observations.

Observations of Comets made at the Royal Observatory, Greenwich, in 1909–1910.

(*Communicated by the Astronomer Royal.*)

The following positions were obtained from photographs taken with the 30-inch reflector and the astrographic 13-inch refractor at the Royal Observatory, Greenwich. The plates were measured with the astrographic micrometer. Six reference stars were, as a rule, measured with the comet, their positions being derived from the catalogues of the Astronomische Gesellschaft. As the mean places of the surrounding stars were used, the positions given are referred to the equator and equinox of 1909·0 or 1910·0, and are automatically corrected for the part of the aberration arising from the Earth's motion. They are not corrected for parallax or the part of the aberration arising from the comet's motion.

The mean places for 1909·0 or 1910·0 may be obtained by applying the corrections for parallax and Earth's aberration, and antedating for the light-time.

Log Parallax Correction = log Parallax Factor − log Δ.

Comet *c* 1909 (Halley).

Date and G.M.T. 1909.				R.A. 1909·0.			Log Parallax Factor.	Dec. 1909·0.			Log Parallax Factor.	Corr. for Earth's Aberration.	
												R.A.	Dec.
	d	h	m	s	h	m	s			′	″	s	″
Sept.	9	15	9	34	6	17	52·45	−9·514	+17	12	2·2	+0·756	−0·43	+2·3
	9	15	45	29	6	17	52·71	9·465	17	12	1·6	0·740	0·43	2·3
	19	15	23	16	6	19	4·37	9·437	17	8	10·5	0·734	0·20	2·3
	23	14	52	41	6	19	1·31	9·463	17	6	34·5	0·740	0·10	2·3
	23	15	54	16	6	19	1·10	9·329	17	6	34·0	0·717	−0·10	2·3
Oct.	6	13	18	57	6	16	10·63	9·517	17	1	33·6	0·758	+0·23	2·2
	6	13	44	0	6	16	10·25	9·485	17	1	33·2	0·747	0·23	2·2
	8	13	18	30	6	15	17·97	9·507	17	0	50·1	0·755	0·29	2·1
	8	13	53	24	6	15	17·28	9·457	17	0	49·1	0·739	0·29	2·1
	19	12	42	24	6	7	46·01	9·488	16	57	14·4	0·749	0·59	2·0
	21	14	10	12	6	5	45·67	9·269	16	56	35·9	0·712	0·64	2·0
	25	12	47	2	6	1	19·10	9·430	16	55	21·8	0·734	0·75	2·0
	25	13	6	14	6	1	18·14	9·391	16	55	21·4	0·727	0·75	2·0
Nov.	5	12	30	20	5	44	2·02	9·328	16	50	33·6	0·719	1·04	2·0
	5	12	52	36	5	44	0·10	9·258	16	50	32·8	0·713	1·04	2·0
	10	12	9	32	5	33	22·76	9·301	16	46	35·0	0·717	1·15	2·1
	10	12	32	29	5	33	20·81	9·221	16	46	36·8	0·711	1·15	2·1
	22	10	56	24	4	59	30·20	−9·274	16	25	58·3	0·718	1·36	2·7
	30	11	54	35	4	29	58·71	+7·221	15	56	11·4	0·706	1·40	3·1
Dec.	1	9	45	6	4	26	20·29	−9·283	15	51	43·0	0·724	1·40	3·2
	3	9	3	24	4	18	16·07	−9·357	+15	41	11·7	+0·733	+1·39	+3·3

Comet *c* 1909 (Halley)—*continued.*

Date and G.M.T. 1909.				R.A. 1909·0.			Log Parallax Factor.	Dec. 1909·0.			Log Parallax Factor.	Corr. for Earth's Aberration.	
												R.A.	Dec.
	d	h	m	h	m	s		°	′	″		s	″
Dec.	8	12	26 36	3	56	17·49	+9·201	+15	8	10·4	+0·725	+1·34	+3·5
	20	6	49 7	3	4	30·49	−9·337	13	26	38·4	0·750	1·04	3·4
	20	7	8 47	3	4	26·97	−9·278	13	26	31·7	0·745	1·04	3·4
	29	8	22 6	2	27	49·07	+8·610	11	57	35·8	0·746	0·70	2·6
	1910.			1910·0.				1910·0.					
Jan.	3	6	37 45	2	10	17·09	−8·843	11	11	39·7	0·754	0·50	2·0
	10	6	23 13	1	48	37·55	−8·113	10	13	16·0	0·761	0·22	0·9
	12	7	31 34	1	43	0·62	+9·072	9	58	9·0	0·767	+0·13	+0·5
	17	6	17 27	1	30	43·67	8·713	9	25	41·0	0·769	−0·05	−0·3
	19	5	48 39	1	26	17·75	8·385	9	14	18·6	0·770	0·12	0·7
	22	6	10 12	1	20	1·95	8·945	8	58	41·6	0·773	0·23	1·2
	26	7	39 11	1	12	27·25	9·394	8	40	50·8	0·789	0·36	1·8
	30	6	37 4	1	5	55·18	9·293	8	26	41·4	0·785	0·48	2·5
Feb.	2	6	44 3	1	1	27·19	9·355	8	17	57·3	0·789	0·57	2·9
	3	7	27 59	1	0	0·63	9·452	8	15	19·7	0·798	0·59	3·1
	11	6	32 49	0	50	11·55	9·429	8	0	42·8	0·797	0·79	4·2
	15	6	43 55	0	45	59·37	9·477	7	56	39·1	0·802	0·88	4·7
	17	6	46 9	0	44	2·20	9·493	7	55	18·8	0·805	0·92	5·0
	17	6	50 22	0	44	2·04	9·499	7	55	18·0	0·806	0·92	5·0
	18	7	1 15	0	43	4·94	9·516	7	54	48·3	0·809	0·94	5·1
	22	7	14 15	0	39	28·96	9·542	7	53	35·5	0·815	1·01	5·6
Mar.	1	7	1 47	0	33	44·01	9·554	7	54	20·2	0·820	1·11	6·3
	5	7	6 4	0	30	37·96	9·563	7	55	51·6	0·825	1·16	6·7
	7	7	12 57	0	29	6·14	+9·567	7	56	50·8	0·828	1·18	6·9
May	6	15	1 1	0	5	9·64	−9·569	9	12	7·5	0·832	0·91	7·9
	6	15	4 17	0	5	10·12	−9·569	9	12	7·9	0·831	0·91	7·9
	6	15	10 21	0	5	10·96	−9·569	9	12	15·0	0·830	0·91	7·9
	22	9	14 54	7	18	50·22	+9·577	14	16	59·2	0·820	0·94	0·2
	23	9	12 52	7	57	5·71	9·566	11	46	33·2	0·813	0·78	0·4
	26	9	10 7	9	4	32·31	9·524	6	30	16·3	0·816	0·47	1·3
June	1	9	39 29	9	55	43·35	9·523	2	1	31·4	0·831	0·31	2·0
	2	9	48 31	10	0	15·52	+9·530	+ 1	37	3·4	+0·832	−0·30	−2·0

Notes.

Sept 9. Very faint.
Oct. 21, Jan. 26, Mar. 7. Very faint and ill-defined.
Nov. 10 (first). Very faint Nov. 10 (second). Sky grew hazy as the exposure proceeded, the star trails in consequence fade towards the latter end.
Dec. 20, Jan. 22. Faint and diffused.
Dec. 29, Jan. 19, Feb. 22. Very diffused.
Feb. 18. Extremely faint and diffused.

Comet *e* 1909 (Daniel).

Date and G.M.T. 1910.				R.A. 1910·0.			Log Parallax Factor.	Dec. 1910·0.			Log Parallax Factor.	Corr. for Earth's Aberration.	
												R.A.	Dec.
	d	h	m	s	h	m	s		°	′	″	s	″
Feb.	3	8	4	7	6	36	49·51	−9·449	+56	32	25·9	−9·570	+2·00	+5·7
	3	8	37	2	6	36	51·21	−9·284	56	32	22·8	9·784	2·00	5·7
	8	10	56	15	6	44	2·39	+9·378	56	31	26·8	9·646	1·90	6·1
	8	11	25	41	6	44	4·24	+9·497	56	13	19·2	9·301	1·90	6·1
	11	8	20	31	6	48	31·42	−9·256	55	57	38·5	9·733	1·83	6·2
	11	8	51	10	6	48	33·33	8·973	55	57	29·6	9·825	1·83	6·2
	15	7	26	33	6	55	4·46	9·472	55	30	53·4	9·160	1·74	6·4
	15	8	7	11	6	55	7·10	−9·277	55	30	40·5	−9·653	1·74	6·4
	15	11	25	53	6	55	20·53	+9·544	55	29	40·8	+8·872	1·74	6·4
	17	8	25	25	6	58	35·92	−9·098	55	15	3·0	−9·716	1·69	6·5
	17	8	49	5	6	58	37·59	8·789	55	14	54·0	−9·768	1·69	6·5
Mar.	1	7	6	41	7	20	48·30	−9·409	53	19	18·6	+8·972	1·41	6·9
	1	7	35	36	7	20	50·43	−9·266	53	19	5·9	−9·033	1·41	6·9
	3	8	29	2	7	24	48·37	−8·537	52	56	4·4	−9·389	1·36	6·9
	7	11	1	38	7	32	53·39	+9·555	52	7	24·4	+9·873	1·26	6·9
	7	11	33	20	7	32	56·01	9·625	52	7	8·4	0·058	1·26	6·9
	14	10	22	8	7	46	49·78	9·467	50	38	41·5	9·836	1·10	6·9
	14	10	50	3	7	46	51·79	9·546	50	38	23·9	9·990	1·10	6·9
	16	9	32	37	7	50	46·35	9·270	50	12	33·7	9·599	1·05	6·9
	16	9	57	53	7	50	48·69	9·387	50	12	14·0	9·757	1·05	6·9
	29	10	47	32	8	16	55·65	9·573	47	9	29·8	0·246	0·78	7·0
	29	11	24	0	8	16	58·75	9·523	47	9	4·4	0·176	0·78	7·0
Apr.	1	10	29	20	8	22	50·62	9·542	46	25	52·0	0·236	0·72	7·0
	1	11	4	29	8	22	53·41	9·608	46	25	28·4	0·333	0·72	7·0
	5	10	7	54	8	30	40·23	9·502	45	26	55·2	0·231	0·65	7·0
	5	10	44	38	8	30	43·14	+9·580	+45	26	33·4	+0·328	+0·65	+7·0

Notes.

March 1 (first), March 29 (first). Touching *réseau.*
March 14. Faint and diffused.
March 16. Extremely faint and diffused.
March 29 (second). Very faint.
April 1. Faint. April 5 (first). Very faint and diffused.
April 5 (second). Touching *réseau* ; faint and diffused.

Comet *a* 1910.

Date and G.M.T. 1910.	R.A. 1910·0.	Log Parallax Factor.	Dec. 1910·0.	Log Parallax Factor.	Corr. for Earth's Aberration. R.A.	Dec.
d h m s	h m s		° ′ ″		s	″
Jan. 28 5 57 39	21 27 35·38	+9·548	−0 6 49·3	+0·836	−1·27	− 5·0
30 5 58 51	21 33 14·64	+9·550	+1 38 7·1	+0·833	−1·27	− 5·1

Comet *b* 1910 (Metcalf).

Date and G.M.T. 1910.	R.A. 1910·0.	Log Parallax Factor.	Dec. 1910·0.	Log Parallax Factor.	Corr. for Earth's Aberration. R.A.	Dec.
d h m s	h m s		° ′ ″		s	″
Aug. 12 10 6 28	16 6 43·46	+9·466	+15 9 52·7	+0·756	+0·35	+10·9
12 10 28 25	16 6 41·45	9·497	15 9 58·1	0·765	0·35	10·9
15 9 53 50	16 0 43·07	9·475	15 30 24·0	0·754	0·24	11·4
15 10 30 4	16 0 40·37	9·521	15 30 34·7	0·770	0·24	11·4
17 9 2 16	15 57 8·77	9·403	15 42 16·0	0·739	0·18	11·5
19 9 0 15	15 53 47·40	9·422	15 53 7·7	0·741	+0·11	11·8
24 9 56 14	15 46 28·34	9·538	16 16 9·0	0·772	−0·05	12·2
26 8 48 19	15 44 2·47	9·468	16 23 40·8	0·747	0·11	12·3
Sept. 9 7 58 25	15 31 57·60	+9·495	+17 4 28·2	+0·750	−0·52	+12·7

Comet *e* 1910 (Faye).

Date and G.M.T. 1910.	R.A. 1910·0.	Log Parallax Factor.	Dec. 1910·0.	Log Parallax Factor.	Corr. for Earth's Aberration. R.A.	Dec.
d h m s	h m s		° ′ ″		s	″
Nov. 28 11 8 5	3 36 43·73	−7·050	+4 41 59·4	+0·805	+1·31	+ 3·7
Dec. 6 9 1 22	3 36 47·74	−9·176	3 44 5·3	0·814	1·27	2·7
19 9 42 25	3 39 50·95	−8·049	3 7 37·9	0·816	1·16	1·0
19 10 0 24	3 39 51·23	+8·236	+3 7 39·4	+0·816	+1·16	+ 1·0

Royal Observatory, Greenwich:
1911 *March* 9.

Measures of Double Stars made at Royal Observatory, Edinburgh.

(*Communicated by the Astronomer Royal for Scotland.*)

The following observations of double stars were made with the 15-inch refractor during the year 1910. The observations were made by Mr. Storey, except for those denoted by an asterisk in the last column, which were made by Mr. Wrigley.

Name of Star.	R.A. 1910. h m	N. Decl. 1910. ° ′	Position Angle. °	Dist. ″	No. of Nights.	Mags.		Epoch 1910+.
β 997	0 0	45 11	337·8	4·00	1	7·9,	8·9	·030
OΣ 547	0 0	45 19	136·4	4·92	1	8·3,	8·3	·030
Σ 3062	0 1	57 56	2·1	1·58	1	6·9,	8·0	·890
Σ 7	0 6	55 28	213·3	1·41	2	8·0,	8·5	·910
Σ 13	0 11	76 27	89·5	0·81	1	6·6,	7·1	·931
Krueger 4 ...	0 14	59 12	189·2	2·26	1	8·5,	9·0	·890
Σ 59	0 42	50 57	148·8	2·18	1	7·2,	8·1	·931
β 1 AB	0 47	56 8	88·0	1·32	1	8·1,	10·1	·049
AC			136·2	4·05	1		8·9	·049
AD			195·8	9·18	1		9·5	·049
OΣ 20	0 49	18 42	311·9	0·50	1	5·9,	7·0	·052
Σ 73	0 50	23 8	28·8	0·89	1	6·2,	6·8	·052
Σ 105	1 12	65 42	184·5	2·99	1	8·5,	9·7	·049
Σ 115	1 17	57 40	143·7	1·15	2	7·3,	7·5	·962
Σ 130	1 28	69 26	186·7	7·55	1	8·0,	9·0	·049
Espin 46	1 32	54 46	33·9	2·65	1	9·0,	10·0	·049
β 870	1 38	57 5	49·0	1·33	1	6·9,	8·3	·994
Σ 204	1 59	69 31	65·4	1·02	1	8·6,	9·1	·052
β 786	2 11	55 20	349·8	5·12	1	8·5,	9·9	·049
Σ 268	2 23	55 8	127·5	2·75	3	6·9,	8·2	·971
Σ 282	2 33	65 16	114·4	7·15	2	8·3,	8·3	·518
Σ 283	2 33	61 6	207·0	1·70	2	8·0,	8·8	·977
Σ 284	2 33	60 54	194·3	5·86	1	8·0,	10·0	·030
Espin 50	2 38	54 33	23·5	2·26	1	9·3,	9·4	·052
Σ 306	2 44	60 3	93·5	2·11	2	7·1,	9·0	·498
Σ 314	2 46	52 38	303·3	1·71	2	6·9,	7·1	·948
Σ 312	2 47	72 32	24·5	3·30	1	7·1,	8·0	·994
Σ 317	2 49	68 50	85·4	3·57	2	7·8,	9·5	·522
Krueger 18 ...	2 58	57 23	274·5	1·50	1	9·2,	9·3	·052
Σ 349	3 3	63 27	320·6	6·25	2	7·4,	8·1	·490
Σ 345	3 4	78 10	84·9	6·69	1	8·0,	9·8	·030

Name of Star.	R.A. 1910.	N. Decl. 1910.	Position Angle.	Dist.	No. of Nights.	Mags.	Epoch 1910+.
	h m	° ′	°	″			
OΣ 52	3 9	65 20	116·8	0·51	1	6·4, 7·0	·052
Σ 344	3 10	84 20	150·5	2·32	1	8·9, 9·7	·030
Hussey 1056 ...	3 11	66 54	283·5	0·54	1	8·1, 8·1	·052
Σ 368	3 13	68 11	160·9	1·95	1	8·5, 8·5	·966
Σ 384	3 21	59 35	269·8	1·90	2	7·8, 9·0	·050
Σ 385	3 21	59 37	163·5	2·42	2	4·7, 9·0	·050
Σ 388	3 22	50 7	208·5	2·95	1	8·2, 9·2	·030
Burnham G.C. 1704	3 22	59 36	51·3	5·47	2	8·6, 10·0	·050
Σ 419	3 33	69 33	75·0	3·17	1	7·2, 7·2	·931
Σ 440	3 40	50 53	229·7	2·59	1	9·2, 9·5	·030
β 743	3 48	52 0	244·8	0 66	1	8·5, 9·0	·052
Bird 1 AB ...	3 54	62 15	223·8	2·06	1	7·2, 8·5	·049
AC ...			173·3	8·72	1	9·5	·049
Σ 460	3 54	80 27	58·5	0·77	2	5·2, 6·1	·509
Σ 480	3 57	55 30	326·4	3·17	3	8·3, 8·5	·694
Σ 490	4 2	59 55	60·5	4·96	2	8·5, 9·0	·550
Σ 503	4 8	63 56	229·2	4·91	1	8·8, 8·8	·112*
Krueger 22 ...	4 10	60 37	189·6	3·65	1	9·3, 9·4	·112*
OΣ 75	4 10	60 17	168·6	0·53	1	7·6, 8·0	·052
Σ 513	4 11	61 22	57·9	5·76	1	7·8, 9·7	·131
Σ 522	4 15	51 24	218·0	1·66	1	8·5, 8·5	·167
Σ 531	4 19	55 26	307·6	0·95	1	7·4, 8·6	·167
β 745	4 21	53 42	129·0	0·47	2	8·3, 8·3	·110
Σ 540	4 22	63 13	179·4	2·87	1	8·3, 10·0	·049
Krueger 23 ...	4 23	55 18	134·4	4·50	1	9·0, 9·3	·178
Krueger 24 ...	4 31	56 45	240·8	3·40	1	9·5, 9·5	·129
Σ 566	4 32	53 18	287·8	1·53	1	5·1, 7·4	·167
Aitken 838 ...	4 33	72 38	10·1	1·13	1	8·7, 9·1	·159
Aitken 1012 ...	4 34	68 55	320·6	1·46	1	8·5, 10·8	·159
Aitken 1013 ...	4 35	59 21	310·8	0·50	1	7·2, 7·2	·167
Σ 608	4 49	51 57	115·1	4·43	1	7·8, 9·2	·049
Krueger 25 ...	4 49	56 30	122·2	2·82	2	9·0, 9·5	·089
A. G. 84	4 55	54 42	160·1	3·70	1	8·8, 9·2	·145*
Σ 558	4 56	86 47	203·9	3·01	1	8·4, 9·9	·088
Σ 625	4 56	58 44	113·9	4·93	1	8·2, 9·8	·129
			118·7	4·74	1		·145*
Σ 635	5 0	54 53	300·3	0·83	1	8·3, 8·3	·052
Hussey 1098 ...	5 4	61 21	116·3	0·89	1	8·9, 9·1	·052
Hussey 1099 ...	5 4	64 38	39·0	0·53	1	8·6, 8·9	·052

Name of Star.	R.A. 1910.	N. Decl. 1910.	Position Angle.	Dist.	No. of Nights.	Mags.	Epoch 1910+.
	h m	° ′	°	″			
Σ 638	5 4	69 44	223·6	4·96	1	7·5, 8·5	·145*
Σ 657	5 11	52 45	288·3	1·31	1	7·5, 8·0	·167
Σ 656	5 12	63 4	214·3	2·65	2	8·3, 10·0	·091
Aitken 845 ...	5 14	74 1	132·6	1·63	1	9·0, 9·8	·173
Σ 676	5 15	64 39	274·6	1·08	2	7·5, 8·5	·170
Σ 677	5 16	63 18	240·6	1·31	2	7·7, 8·0	·170
Σ 689	5 19	67 50	324·9	6·10	1	8·0, 10·0	·030
Hussey 1105 ...	5 21	62 38	240·8	2·78	2	8·5, 10·0	·127
Σ 695 AB ...	5 27	79 17	158·2	10·64	1	8·3, 9·0	·058
BC ...			174·0	2·12	1	9·7	·058
Σ 723	5 27	51 52	108·3	4·68	1	8·4, 10·5	·131
Σ 739	5 34	66 30	250·3	2·29	1	8·3, 9·3	·049
Σ 769	5 38	53 17	172·2	3·99	1	8·0, 10·2	·145*
Σ 3115	5 39	62 47	17·6	1·33	2	6·7, 7·8	·114
Σ 780 AB ...	5 41	65 44	103·6	3·91	3	6·7, 7·9	·069
AC ...			151·6	11·87	2	10·2	·040
AB ...			107·6	3·93	1	7·9	·071*
Hussey 1113 ...	5 44	60 50	251·1	1·40	2	9·2, 9·4	·128
Σ 810	5 49	52 55	246·1	2·67	1	8·8, 9·5	·049
Burnham G.C. 3059	5 52	52 41	306·1	1·74	1	9·0, 9·3	·058
Hussey 1116 ...	5 52	63 37	311·3	1·44	1	8·8, 9·2	·055
OΣ 121	5 53	74 0	166·7	0·52	2	7·3, 8·5	·053
Σ 784	5 56	84 13	197·4	1·36	2	8·7, 8·7	·129
			195·9	1·63	1		·071*
Σ 881	6 14	59 25	110·9	0·83	3	6·4, 7·9	·155
Hussey 828 ...	6 15	82 36	108·0	1·15	1	8·8, 9·0	·049
Σ 918	6 26	52 32	328·5	4·40	2	6·7, 7·7	·128*
Σ 925	6 30	67 25	92·0	3·45	1	7·8, 10·3	·131
Σ 935	6 31	52 23	323·1	3·83	2	8·2, 9·0	·127
Σ 934	6 31	55 7	332·1	4·33	2	8·7, 9·5	·127
Σ 936	6 31	58 11	271·6	1·44	3	7·0, 8·7	·159
Σ 948 AB ...	6 38	59 32	116·4	1·58	2	5·2, 6·1	·201
AC ...			306·8	8·80	2	7·4	·201
Σ 958	6 40	55 48	256·6	5·05	1	6·0, 6·0	·167
Σ 963	6 45	59 33	88·5	0·47	1	5·9, 7·1	·055
OΣ 159	6 49	58 33	14·8	0·86	4	5·1, 6·2	·189
Σ 980	6 54	72 48	190·2	3·52	2	8·6, 10·1	·149
Σ 1001 AB ...	6 55	54 19	65·3	9·04	2	7·1, 8·7	·135
BC ...			0·3	1·76	3	9·0	·146

Name of Star.	R.A. 1910.	N. Decl. 1910.	Position Angle.	Dist.	No. of Nights.	Mags.	Epoch 1910+.
	h m	° ′	°	″			
Σ 1009	6 58	52 54	153·2	3·80	2	6·7, 6·8	·149
Hussey 618 ...	7 5	51 32	119·8	1·34	1	8·8, 10·8	·216
Σ 1039 ...	7 10	63 41	211·7	3·24	1	8·8, 9·5	·123
Σ 1055	7 13	60 4	328·6	2·09	3	6·0, 10·5	·221
Σ 1051 AB ..	7 15	73 16	281·0	1·05	3	6·5, 8·6	·143
AC ...			82·5	31·60	2	6·7	·159
Σ 1093	7 23	50 11	143·9	0·71	2	8·2, 8·2	·242
Hussey 622 ...	7 30	50 48	33·9	2·89	2	9·0, 9·8	·195
Σ 1107	7 35	76 1	203·6	1·34	2	8·3, 10·2	·224
Σ 1131	7 42	71 41	179·0	2·61	2	9·3, 9·5	·172
A.G. 146 ...	7 51	50 30	286·6	3·32	2	9·1, 9·1	·170
Σ 1151	7 53	77 2	221·5	3·84	2	8·7, 10·2	·176
Σ 1172	7 57	55 0	246·6	1·63	3	7·6, 9·4	·198
Espin 71 ...	8 6	53 32	285·9	3·35	2	9·0, 9·1	·156
Σ 1192 AB ...	8 8	60 40	259·3	2·81	2	6·8, 10·5	·196
AC ...			223·3	49·13	2	10·2	·196
Σ 1205	8 12	56 44	175·5	1·29	2	8·5, 8·8	·155
Hussey 854 ...	8 16	65 46	213·5	1·47	1	9·2, 9·5	·151
Σ 1225	8 23	51 30	191·1	3·71	3	8·5, 8·5	·201
Σ 1235	8 27	57 14	83·7	1·15	1	8·0, 10·0	·200
Krueger 30 ...	8 39	58 2	120·8	2·01	1	9·5, 9·5	·145
Σ 1280	8 47	71 9	45·7	4·85	2	7·5, 7·6	·193
β 408	8 51	63 46	346·1	3·00	2	7·8, 10·3	·156
Σ 1284	8 54	81 24	167·7	2·59	1	8·0, 9·7	·159
Σ 1303	9 0	65 21	279·9	2·87	2	8·3, 10·2	·163
Σ 1313	9 5	70 21	264·0	1·09	3	8·5, 8·7	·181
Σ 1331 AB ...	9 13	61 44	153·1	0·88	3	8·0, 8·0	·238
AC ...			190·5	12·38	3	11·5	·238
OΣ 200	9 18	51 57	335·8	1·51	2	6·7, 8·4	·220
Σ 1345	9 19	64 44	85·6	2·73	2	8·5, 10·1	·231
Σ 1359	9 26	56 39	60·3	7·67	1	8·5, 9·2	·246
Hussey 565 ...	9 29	50 28	184·4	1·72	1	8·8, 8·8	·216
Σ 1362	9 29	73 29	131·0	5·29	1	7·0, 7·0	·266*
Σ 1373	9 36	77 8	131·6	1·82	1	8·2, 9·5	·200
Hussey 630 ...	9 41	51 23	72·4	2·28	2	8·8, 8·8	·231
Krueger 33 ...	9 44	58 38	214·8	2·15	1	9·0, 9·0	·233
Σ 1381	9 44	61 3	211·6	1·48	2	8·5, 8·7	·216
Σ 1387	9 47	69 22	271·2	9·30	1	9·5, 9·5	·216
Σ 1386	9 47	69 20	293·1	2·12	2	8·2, 8·2	·224

Name of Star.	R.A. 1910.	N. Decl. 1910.	Position Angle.	Dist.	No. of Nights.	Mags.		Epoch 1910+.
	h m	° ′	°	″				
Σ 1400	9 55	69 13	229·7	2·53	3	7·3,	10·5	·238
Σ 1408	10 3	73 29	12·1	3·46	2	8·4,	9·2	·230
Aitken 1086 ...	10 20	67 50	212·4	1·44	2	8·5,	9·5	·234
Σ 1428	10 20	53 5	86·3	3·19	2	7·5,	7·8	·266
Σ 1499	11 1	83 35	315·0	7·26	2	8·5,	9·3	·240
Σ 1510	11 2	53 18	336·5	4·58	2	7·1,	8·4	·264
A. G. 174 ...	11 17	65 13	105·9	2·22	1	10·0,	10·3	·233
OΣ 235	11 27	61 35	231·9	0·55	3	6·0,	7·3	·242
Krueger 38 ...	11 27	60 34	50·3	2·55	1	9·0,	9·2	·236
Σ 1559	11 32	64 51	322·7	1·90	1	6·7,	7·7	·244
Hussey 728 ...	11 33	50 18	106·9	0·46	2	7·5,	8·5	·245
Aitken 74 ...	11 36	71 27	315·0	1·27	1	9·1,	9·5	·246
Σ 1860 ...	14 31	55 36	100·2	1·46	1	7·5,	8·7	·720
Σ 1872	14 38	58 21	43·0	7·96	2	7·0,	8·0	·743
Σ 1871 ...	14 38	51 46	294·3	2·01	2	7·0,	7·0	·705
Σ 1898	14 54	59 45	212·3	2·85	2	7·8,	9·8	·713
Σ 1905	14 56	71 12	159·2	3·56	2	8·3,	8·3	·695
Σ 1928	15 9	72 47	276·7	6·70	1	8·5,	9·2	·767
Σ 3125	15 24	67 23	269·6	2·20	1	8·7,	9·0	·720
Aitken 1124 ...	15 33	55 49	138·3	1·24	2	8·6,	9·3	·736
Σ 1969	15 39	60 16	59·2	0·45	1	8·0,	8·7	·706
Σ 2006	15 58	59 11	192·2	1·59	2	7·5,	9·2	·705
Σ 2045	16 18	61 43	181·0	2·47	2	8·0,	9·2	·705
Σ 2054	16 22	61 54	359·4	1·33	3	5·7,	6·9	·756
OΣ 312	16 22	61 43	140·7	5·24	3	2·1,	8·1	·676
Burnham G.C. 7664	16 26	60 53	75·1	1·43	2	8·0,	8·5	·712
Espin 76	16 41	50 47	45·4	2·36	1	9·0,	9·5	·761
Σ 2130	17 3	54 36	137·3	2·51	3	5·0,	5·1	·763
Σ 2153	17 15	49 24	263·9	1·88	1	8·6,	9·1	·783
Krueger 46 ...	17 18	58 37	64·1	1·59	1	8·8,	9·0	·624
Hussey 672 ...	17 23	51 34	59·9	4·58	1	8·0,	11·0	·720
β 1201	17 26	67 51	341·0	0·60	2	7·8,	7·8	·705
Σ 2180	17 26	50 57	261·2	3·62	2	7·0,	7·2	·712
β 962	17 34	61 57	346·4	1·23	2	5·5,	10·1	·733
Σ 2199	17 36	55 48	83·4	1·92	3	7·2,	7·8	·709
Hough 559 ...	17 37	63 27	301·5	4·40	3	9·0,	10·0	·695
Σ 2207	17 37	67 10	125·2	0·98	4	8·0,	8·5	·744
Σ 2218	17 39	63 43	345·2	2·16	2	6·5,	7·7	·762
Σ 2210	17 39	49 2	121·9	3·15	2	8·5,	10·0	·773

Name of Star.	R.A. 1910.	N. Decl. 1910.	Position Angle.	Dist.	No. of Nights.	Mags.	Epoch 1910+.
	h m	° ′	°	″			
Hussey 925 ...	17 55	67 29	327·8	1·89	1	9·0, 10·5	·706
Σ 2284	18 1	65 57	193·7	3·58	1	7·6, 9·2	·682
OΣ 344	18 4	49 42	144·2	2·15	2	6·7, 10·8	·765
Hussey 674 ...	18 7	50 23	272·2	0·67	2	7·5, 8·0	·736
Hussey 926 ...	18 9	64 13	267·8	4·14	2	9·1, 9·3	·723
Σ 2307	18 12	69 13	204·0	4·44	1	8·5, 8·5	·720
Σ 2368	18 36	52 16	326·8	1·90	2	7·2, 7·4	·701
Σ 2384	18 38	67 2	306·2	0·90	1	8·0, 8·5	·706
Hussey 755 ...	18 42	51 55	121·5	0·61	1	8·7, 9·0	·706
Σ 2403	18 43	60 57	263·9	1·65	1	6·2, 9·0	·706
Σ 2410	18 45	59 14	92·7	1·83	2	8·2, 8·7	·740
Σ 2451	19 0	51 28	70·7	2·24	2	8·7, 9·0	·684
Σ 2478	19 2	69 19	298·4	1·27	1	8·8, 8·8	·704
Δ 19 AB ...	19 6	55 11	13·6	0·52	1	7·0, 8·0	·687
AB & C ...			30·3	6·90	1	9·4	·687
OΣ 369	19 8	71 56	32·0	0·91	2	7·0, 7·3	·732
Σ 2509	19 16	63 2	332·9	1·41	2	7·0, 8·1	·695
Σ 2550	19 29	73 10	68·2	2·16	1	8·2, 8·2	·742
Σ 2553	19 32	61 51	103·8	1·13	1	8·4, 9·2	·783
Σ 2555	19 34	53 10	279·6	1·93	2	8·5, 9·1	·712
Küstner 2 ...	19 35	71 24	267·4	1·38	1	7·2, 9·2	·783
Σ 2564	19 36	63 37	169·9	9·92	2	8·5, 10·2	·773
Espin	19 38	64 43	311·7	3·15	1	8·8, 9·4	·704
Σ 2603	19 48	70 3	4·6	3·17	1	4·0, 7·6	·682
Σ 2605	19 53	52 12	180·5	3·25	2	5·0, 7·5	·717
Hussey 955 ...	19 56	63 12	334·7	3·01	2	9·0, 11·2	·754
Aitken 865 ...	20 1	70 12	89·9	2·70	1	8·0, 10·4	·742
Σ 2640	20 3	63 38	20·1	5·78	1	6·0, 9·9	·742
β 470	20 4	63 30	218·8	2·94	1	9·5, 11·0	·742
Σ 2642	20 4	63 26	177·7	2·49	1	8·7, 8·7	·742
Σ 2652	20 7	61 48	258·9	0·63	1	7·3, 7·6	·682
Σ 2675	20 11	77 26	122·1	7·57	2	4·0, 8·0	·733
Σ 2671	20 16	55 7	337·9	3·38	2	6·0, 7·4	·753
Aitken 869 ...	20 22	75 48	271·0	2·14	2	9·1, 9·5	·732
Aitken 739 ...	20 30	56 53	197·4	4·41	1	9·0, 11·7	·704
Σ 2717	20 35	60 26	259·9	2·28	1	7·2, 9·7	·745
β 471	20 41	62 6	307·1	1·91	1	10·0, 10·0	·682
β 366	20 45	50 10	131·3	1·59	2	8·2, 8·5	·764
Σ 2732	20 45	51 35	68·6	4·25	1	6·7, 8·7	·745

Name of Star.	R.A. 1910. h m	N. Decl. 1910. ° ′	Position Angle.	Dist. ″	No. of Nights.	Mags.		Epoch 1910+.
β 155	20 48	51 5	31·4	0·83	2	6·5,	7·4	·732
Aitken 756 ...	20 55	58 28	228·8	0·59	1	7·3,	8·0	·890
Σ 2741	20 55	50 7	30·8	2·16	3	6·0,	7·3	·766
β 68	20 56	49 52	149·2	1 89	3	8·5,	9·2	·778
Σ 2751	20 59	56 19	346·0	2 17	2	6·0,	7·0	·880
Σ 2771	21 3	70 24	216·0	2·96	1	8·8,	8·8	·871
Σ 2783	21 11	57 55	32·3	1·06	1	8·0,	8·0	·887
Σ 2790	21 16	58 15	43·7	4·56	1	5·6,	9·9	·871
Σ 2789	21 17	52 36	116·6	6·40	2	7·1,	7·1	·746
Σ 2801	21 21	79 58	273·0	2·01	1	7·3,	8·0	·890
Σ 2807	21 22	82 8	315·9	2·31	1	8·2,	8·3	·890
Hussey 593 ...	21 27	49 44	59·0	1·72	1	8·8,	9·0	·887
Σ 2837	21 41	82 31	294·1	2·96	2	8·5,	9·0	·880
Σ 2827	21 41	63 11	205·3	5·17	1	8·5,	9·0	·871
Σ 2843	21 49	65 19	139·5	2·08	1	7·0,	7·2	·931
OΣ 456	21 52	52 6	30·7	1·44	2	7·8,	8·0	·910
Σ 2860	22 0	60 25	253·0	8·28	1	7·7,	9·3	·931
Σ 2863	22 1	64 11	282·7	7·32	1	4·7,	6·5	·931
Krueger 58 ...	22 17	59 25	34·5	2·09	1	9·0,	9·1	·871
Σ 2947	22 45	68 5	65·4	4·23	1	7·2,	7·2	·890
Σ 2950	22 47	61 13	306·4	2·23	1	5·7,	7·0	·931
OΣ 490	23 6	56 58	302·4	1·29	1	7·2,	9·2	·890
Aitken 787 ...	23 10	68 36	199·4	2·90	1	9·3,	9·4	·890
Σ 3001	23 14	67 37	204·6	2·72	1	5·2,	7·8	·931
β 857	23 36	67 3	299·1	1·27	1	8·5,	8·9	·890
Σ 3049	23 54	55 15	326·6	3·78	1	5·4,	7·5	·931

The Long-Period Variable RT Cygni (*Ch.* 7085) *in* 1910.

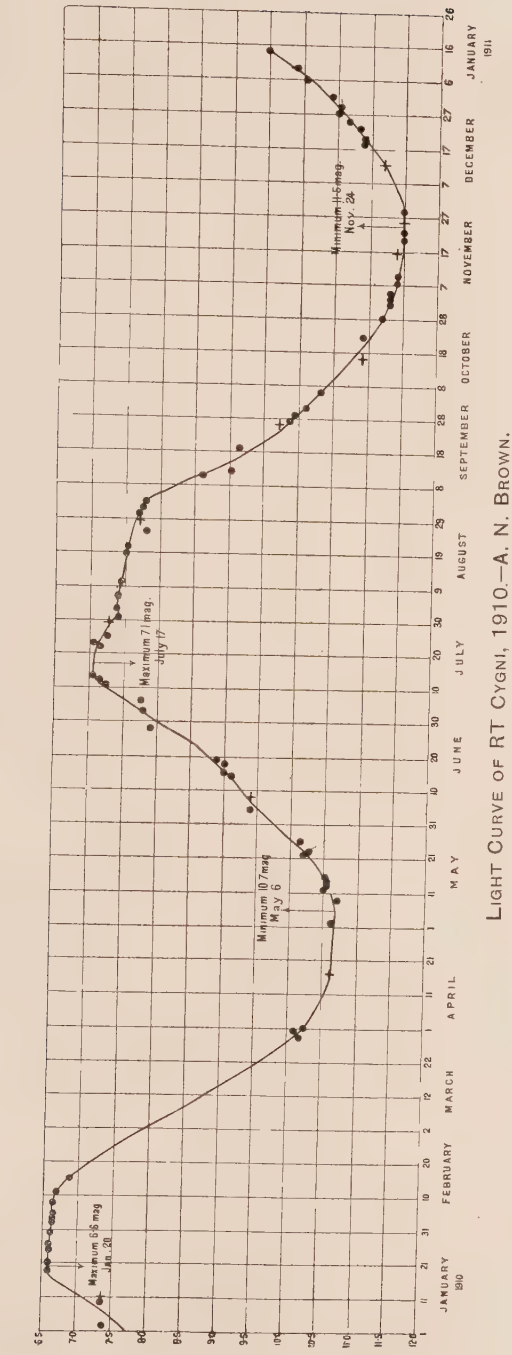
By A. N. Brown, M.A. (Plate 11.)

The following observations, 86 in number, are in continuation of the series published last year in *Monthly Notices*, vol. lxx., No. 3. They reveal two maxima and two minima, as shown on the light-curve appended.

In the first column the date is given, and in the second the corresponding Julian date, omitting the constant number 2,410,000. In the third column the letters T, t, and p indicate respectively refractors of 5-in., 3-in., and 2-in. aperture, each of these letters being followed by the magnifying power used. The fourth column gives the value attached to each observation, 1 signifying good, reliable; 2 fairly good—the moon, clouds, etc., having caused some difficulty. The comparisons in the fifth column were for the most part made by the "fractional" method, but each is here expressed in steps, the value of a step in every case being ·1 magnitude. Sometimes, however, fractional and step estimates were made upon the same occasion. In these cases the different estimates are separated by a semicolon, which is also used to divide different fractional estimates from each other. Thus on 1910 June 3 one fractional followed by one step observation is shown; on September 29 there are two fractional followed by one step estimate; on 1911 January 6 two step estimates were made. The two methods are thus clearly distinguished from each other. All the comparison stars used are numbered as in Hagen's Atlas, the negative sign after the comparison star implying that the variable was fainter, the positive sign that it was brighter, than the star in question. As regards the magnitude of each comparison star, the Harvard value, where this was forthcoming, has been adopted; when the H.C.O. does not assign a value to one of Hagen's stars used, Hagen's own value, as given in the *Atlas Stell. Var.*, has been taken. In the last column the magnitude deduced from the comparisons is given, a mean being taken when two or more separate estimates have been made. In such cases fractional and step estimates are given equal weight.

In the curve a dot shows a first class, a cross a second class observation.

[*The*



LIGHT CURVE OF RT CYGNI, 1910.—A. N. BROWN.

The Observations.

Date. 1910.	Julian date.	Instr.	Class.	Comparisons.	Ded. mag.
Jan. 3	8675	t. 30	1	3 − 7·5, 4 + 2·5	7·4
10	8682	,,	,,	3 − 7, 4 + 3	7·4
11	8683	,,	2	3 − 7, 4 + 3	7·4
19	8691	,,	1	2 − 0·5, 3 + 1	6·6
21	8693	T. 25	,,	2 − 0·5, 3 + 1	6·6
25	8697	,,	,,	2 − 1, 3 + 0·5	6·6
26	8698	t. 30	,,	2 − 1, 3 + 1	6·6
30	8702	,,	,,	2 − 1, 3 + 0·5	6·6
Feb. 2	8705	T. 25	,,	2 − 1·5, 3 + 0·5	6·6
3	8706	t. 30	,,	2 − 1·5, 3 + 0·5	6·6
4	8707	T. 25	,,	2 − 1·5, 3 + 0·5	6·6
8	8711	,,	,,	2 − 1·5, 3 + 0·5	6·6
11	8714	,,	,,	= 3	6·7
15	8718	,,	,,	3 − 2, 4 + 8	6·9
Mar. 29	8760	T. 120	,,	22 − 7, 41 + 2	10·2
31	8762	,,	,,	25 − 3·5, 41 + 3·5	10·1
Apr. 1	8763	,,	,,	22 − 7·5, 41 + 1·5	10·3
17	8779	t. 150	2	= 43	10·6
May 2	8794	T. 120	1	= 43	10·6
9	8801	T. 200	,,	43 − 1, 44 + 2	10·7
12	8804	T. 120	,,	43 + 1	10·5
13	8805	,,	,,	43 − 1·5, 43 + 0·5	10·6
14	8806	T. 150	,,	41 − 1, 43 + 0·5	10·5
15	8807	T. 120	,,	41 − 1, 43 + 0·5	10·5
22	8814	T. 95	,,	30 − 2, 41 + 2	10·2
23	8815	,,	,,	30 − 2·5, 41 + 1·5	10·3
26	8818	,,	,,	30 − 1·5, 41 + 2·5	10·2
June 3	8826	T. 50	,,	19 − 1, 22 + 1 ; 23 + 1·5	9·4
8	8831	t. 75	2	22 + 1	9·4
14	8837	T. 50	1	16 − 4, 19 + 2 ; 15 − 3, 20 + 1·5	9·1
15	8838	,,	,,	16 − 3, 19 + 3 ; 15 − 1·5, 20 + 3·5	9·0
18	8841	,,	,,	16 − 3, 19 + 3 ; 15 − 2, 20 + 3	9·0
19	8842	,,	,,	16 − 1·5, 19 + 4·5 ; 15 − 1·5, 20 + 3·5	8·9
28	8851	,,	,,	5 − 1, 7 + 0·5	7·9
July 3	8856	t. 30	1	4 − 1·5, = 5 ; 4 − 2, 6 + 1	7·8
6	8859	,,	,,	4 − 1·5, 6 + 1·5	7·8
11	8864	,,	,,	3 − 6, 4 + 4	7·3
12	8865	T. 25	,,	3 − 5, 4 + 5	7·2

The Observations—continued.

Date.	Julian date.	Instr.	Class.	Comparisons.	Ded. mag.
1910.					
July 13	8866	t. 30	1	3 − 4, 4 + 6	7·1
22	8875	,,	,,	3 − 5, 4 + 5	7·2
23	8876	,,	,,	3 − 4, 4 + 6	7·1
25	8878	T. 25	,,	3 − 6, 4 + 4	7·3
29	8882	t. 30	2	3 − 6, 4 + 4	7·3
31	8884	,,	1	3 − 7, 4 + 2	7·4
Aug. 2	8886	,,	,,	3 − 7, 4 + 2·5	7·4
6	8890	,,	,,	3 − 7, 4 + 2	7·4
10	8894	,,	,,	3 − 8, 4 + 2	7·5
19	8903	p. 26	,,	3 − 8·5, 4 + 1	7·5
20	8904	p. 16	,,	4 + 1	7·5
25	8909	p. 26	,,	4 − 1·5, 6 + 1	7·8
29	8913	t. 30	2	4 − 1·5, 6 + 1 ; 5 + 2	7·7
30	8914	,,	1	4 − 1·5. 6 + 1 ; 5 + 2	7·7
Sept. 1	8916	,,	,,	4 − 2, 6 + 1 ; 5 + 1	7·8
3	8918	,,	,,	4 − 2·5, 6 + 0·5 ; 5 + 1	7·8
11	8926	,,	,,	10 − 1, 13 + 0·5 ; 12 − 2, 15 + 1	8·6
12	8927	,,	,,	16 − 2·5, 19 + 3·5 ; 15 − 3, 20 + 2	9·0
19	8934	T. 50	,,	16 − 4, 19 + 2 ; 15 − 4, 20 + 1	9·1
26	8941	t. 75	2	22 − 2	9·7
27	8942	T. 120	1	22 − 3·5, 41 + 5·5 ; 25 − 1·5	9·9
29	8944	T. 50	,,	25 − 2, 30 + 1 ; 25 − 2, 41 + 5 ; 27 + 2	9·9
Oct. 1	8946	T. 120	,,	30 − 1, 41 + 3	10·1
6	8951	,,	1	41 + 1	10·3
16	8961	T. 150	2	43 − 3, 44 + 1	10·9
22	8967	T. 200	1	43 − 3, 44 + 1	10·9
28	8973	,,	,,	44 − 2	11·2
Nov. 1	8977	,,	,,	44 − 3	11·3
2	8978	,,	,,	44 − 3	11·3
4	8980	,,	2	44 − 3	11·3
7	8983	,,	1	44 − 4	11·4
9	8985	,,	,,	44 − 4	11·4
16	8992	,,	2	44 − 4	11·4
20	8996	,,	1	44 − 5	11·5
22	8998	T. 250	,,	44 − 5	11·5
25	9001	T. 300	2	44 − 5	11·5
28	9004	T. 167	1	44 − 5	11·5
Dec. 12	9018	T. 150	2	44 − 2	11·2

The Observations—continued.

Date	Julian date.	Instr.	Class.	Comparisons.	Deg. mag.
1910.					
Dec. 18	9024	T. 167	1	43 − 3 44 + 1	10·9
19	9025	T. 150	,,	43 − 3, 44 + 1	10·9
22	9028	T. 200	,,	43 − 2, 44 + 1·5	10·8
24	9030	t. 150	,,	43 − 0·5, 44 + 3	10·7
27	9033	t. 120	,,	43 + 1	10·5
29	9035	,,	,,	41 − 1·5, 43 + 0·5	10·6
1911.					
Jan. 1	9038	t. 120	1	41 − 0·5, 43 + 2	10·4
6	9043	t. 85	,,	= 30 ; 41 + 3	10·1
9	9046	t. 75	,,	25 − 2, 30 + 1·5	9·9
14	9051	,,	,,	= 22 ; = 23	9·5

The Comparison Stars used.

Hagen's number.	Magnitude. H.P.	Magnitude. Hagen.	Hagen's number.	Magnitude. H.P.	Magnitude. Hagen.
2	6·50		19		9·3
3	6·67		20		9·3
4	7·64		22		9·5
5		7·8	23		9·5
6		7·9	25		9·7
7		8·0	27	9·84	
10	8·39		30		10·0
12		8·5	41		10·4
13	8·54		43		10·6
15		8·8	44		11·0
16	8·71				

Ludgrove, New Barnet, Herts:
1911 *January* 24.

Report of the University Observatory, Oxford, for 1910.
(*Director, Professor H. H. Turner.*)

The work of the year has been the preparation of vol. vii. of the *Oxford Astrographic Catalogue*, including the repetition of a large number of the early plates (taken in 1892 and following years) which did not come up to later standards. The measures of the early plates were, however, not wasted; they have been compared with the later measures for the detection of proper motions, in a manner similar to that already used in other zones. In zone $+25^\circ$ no less than 91 plates (more than half the zone) are available for comparison with plates taken ten years later, and the total number of stars compared is 17,000, which is about equal to the total for all the other zones. The work of examining the cases of discordance, eliminating mere mistakes, and confirming cases of real motion has been very steadily carried on by Mr. W. H. Waterfield of Christ Church, and is now nearly complete. A grant from the Government Grant Fund rendered possible the straightforward numerical work of comparison.

Owing to the number of new plates to be taken and measured the printing of vol. vii. occupied the whole year, but it will very shortly be ready for distribution. The next item in the programme is the collection of various discussions arising out of the work, which are to form volume viii. One of these is the definitive examination of the residual errors of the places of the reference stars, including the discussion of "Magnitude Equation" in the meridian observations. And another will, of course, be the collation of the proper motions deduced by the comparison of plates.

Arrangements are being made to determine the places of the reference stars by photography in at least one zone, in order to test the practicability of this method.

Early in the year Mr. Plummer studied the theory of Stellar Aberration and the Principle of Relativity (*Mon. Not.*, lxx. p. 252), and also the Formulæ of Interpolation (*ibid.*, p. 266). During the year the optical distortion of the plates forming the "Harvard Sky" was examined, a number of measures being made for that purpose by Mr. G. H. Hamilton (*Mon. Not.*, lxxi. p. 106).

The Director attended the meeting of the Union for Solar Research on Mount Wilson in September last.

[It will be seen that the statements on pp. 357–8 of the Council Report with regard to the work of this observatory are not quite correct, vol. vii. having been overlooked.]

Erratum in Annual Report (*Note on the Eötvös Torsion Balance*).

Page 360, line 10 from bottom, *for* $\frac{d^2u}{dx^2}$ *read* $\frac{d^2u}{dz^2}$.